HOPF MAXIMUM PRINCIPLE REVISITED

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Abstract. A weak version of Hopf maximum principle for elliptic equations in divergence form \( \sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u) = 0 \) with Hölder continuous coefficients \( a_{ij} \) was shown in [3], in the two-dimensional case. It was also pointed out that this result could be extended to any dimension. The objective of the present note is to provide a complete proof of this fact, and to cover operators more general than the one studied in [3].

1. Introduction

It is well-known that the Hopf maximum principle (see [5, Lemma 3.4], [10, Theorem II.7] or [11, Theorem 2.8.4] for a classical statement) does not hold for linear elliptic equations in divergence form. More precisely, a function \( u \in C^1(\Omega) \), with \( \Omega \subset \mathbb{R}^N \) a smooth domain, is assumed to solve in weak sense the elliptic equation

\[
\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) = 0 \tag{1.1}
\]

in \( \Omega \), while \( u(x) > u(x_0) \) in an inner tangent ball \( B \subset \Omega \), \( x_0 \in \partial \Omega \cap \overline{B} \) being the tangency point. Then, a maximum Hopf principle (a "boundary point lemma") holds at \( x_0 \) if the strict inequality

\[
\frac{\partial u}{\partial n} < 0 \tag{1.2}
\]

is satisfied at \( x_0 \), \( n \) standing for the outward unit normal at that point.

A counterexample to this assertion, even when coefficients \( a_{ij} \) in (1.1) are continuous in \( \overline{\Omega} \) was given in [5, Problem 3.9] (see also [11, Section 2.7]; and a further example in [9] for the case in which coefficients in (1.1) satisfy \( a_{ij} \in L^\infty(\Omega) \)). Moreover, as pointed out in [9], a simpler example than the one in [5] can be obtained as follows. Function \( u = \frac{z}{|z|^2} \), \( z = x + iy \), is harmonic and negative in the plane domain \( \Omega \) enclosed by the \( C^1 \) curve \( r = \varphi(\theta) \) with \( \varphi(\theta) = \exp (-\theta \tan \theta) \) if \( |\theta| < \frac{\pi}{2} \), \( \varphi(\pm \pi/2) = 0 \) ([5, Chapter 3]). Outward unit normal at \((0,0) \in \partial \Omega \) is \( n = (-1,0) \) while

\[
u(0,0) = 0, \quad u_x(0,0) = 0.
\]

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Thus (1.2) fails. Since \( \Omega \) is not a \( C^2 \) domain at \((0, 0)\) then, after a \( C^1 \) rectification of \( \partial \Omega \) near \((0, 0)\) one finds that \( u \) solves an equation (1.1) with respect to new variables \((x', y')\) in \( B \cap \{ y' > 0 \} \), with coefficients \( a_{ij} \in C(B \cap \{ y' \geq 0 \}) \), being \( B \) a small ball centered at \((0, 0)\). This furnishes us the desired counterexample.

Nevertheless, Hopf maximum principle, when regarded in this weak form, seems to be either not correctly stated (see e. g. [1, Proposition 1.16] where some kind of differentiability assumption on the coefficients seems to be missing) or not properly employed in comparison arguments (proof of [6, Proposition 2.2], Remark 2.1 below).

The difficulty in showing a Hopf maximum principle for (1.2) lies, of course, on the lack of differentiability of coefficients \( a_{ij} \). Indeed, a proof in the line of the standard one works provided that the \( a_{ij} \) belong to \( C^0(B \cap \{ y' \geq 0 \}) \). That is why it still seems an outstanding result the fact that Hopf principle holds when the \( a_{ij} \) are merely Hölder continuous. This was shown in [3, Lemma 7] for (1.1) in the case \( N = 2 \) (a later improved two-dimensional statement appeared in [4]). Moreover, it is asserted in [3, Remark 2 p. 35] that: “The proof of Lemma 7 can be extended to \( n \) dimensions for equations of the form (1.1)”. Accordingly, the goal of this note is to furnish to the interested reader a detailed proof of such extended version. In addition, the operators we are addressing in the present article are slightly more general than the one announced in [3], meanwhile some of the auxiliary results obtained here seem interesting in its own right (see estimate (3.8) in Lemma 3.4 below).

To simplify notation we are using, whenever possible, either \( \partial_i \) or \( \partial_{ij} \) instead of \( \partial_\partial x_i \) or \( \partial_\partial x_i \partial_\partial x_j \), respectively, wherein reference variable \( x \) could be replaced by another one, say \( y \) depending on the context.

Our main result is stated as follows.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^N \) be a smooth bounded domain, \( a_{ij} \in C^\alpha(\bar{\Omega}) \) with \( a_{ij}(x) = a_{ji}(x) \), \( 1 \leq i, j \leq N \), \( x \in \Omega \), and

\[
\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j > 0,
\]

for all \( x \in \bar{\Omega} \) and \( \xi \in \mathbb{R}^N \setminus \{0\} \). Assume that \( u \in C^1(\bar{\Omega}) \) solves, in the weak sense,

\[
-\sum_{i,j=1}^{N} \partial_i(a_{ij}(x) \partial_j u) + \sum_{i=1}^{N} b_i(x) \partial_i u + c(x) u \geq 0,
\]

in \( \Omega \), where \( b_i \in L^\infty(\Omega) \) for \( 1 \leq i \leq N \), \( c \in L^\infty(\Omega) \) and \( c(x) \geq 0 \) a. e. in \( \Omega \).

Suppose that for \( x_0 \in \partial \Omega \) there exists a ball \( B \subset \Omega \) with \( x_0 \in \partial B \) where \( u = u(x) \) satisfies:

\[
u \rangle > u(x_0)
\]

in \( B \).

If \( u(x_0) \leq 0 \) then

\[
\partial_{\nu} u(x_0) < 0,
\]

where \( \nu \) is any outward direction, i. e. any unitary vector \( \nu \in \mathbb{R}^N \) so that \( \langle \nu, n \rangle > 0 \), \( n \) being the outward unitary normal at \( x_0 \).

**Remark 1.2.** (a) Ball \( B \) in the statement is indeed an “inner ball” tangent to \( \partial \Omega \) at \( x_0 \in \partial \Omega \).
(b) No restriction on the sign of \( c \) is required in the case where \( u(x_0) = 0 \). Alternatively, the sign of \( u(x_0) \) can be arbitrary provided that \( c(x) = 0 \) for all \( x \in \Omega \).

2. **Proof of Theorem 1.1**

Define the operator,
\[
L u = - \sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u),
\]
which is understood to act in weak sense on functions in \( C^1(\Omega) \). Let \( u \) be as in the statement of Theorem 1.1 and set \( u_0 = u(x_0) \). Then
\[
L(u - u_0) + \sum_{i=1}^{N} b_i \partial_i (u - u_0) + c(u - u_0) \geq -cu_0 \geq 0.
\]

By performing a suitable linear transformation of the variable \( x \) it can be assumed that \( a_{ij}(x_0) = \delta_{ij} \) (\( \delta_{ij} = 1 \) if \( i = j \), \( \delta_{ij} = 0 \) otherwise). After a translation and a rotation it can be also assumed that \( x_0 = 0 \) meanwhile the outward unit normal \( n \) at \( x = 0 \) is \(-e_N\). It should be remarked that after this set of variable changes, outward derivatives of \( u \) at \( x_0 \) are transformed into outward derivatives of \( u \) at 0, with respect to the new variables.

Consider the “unitary” annulus \( D = \{ x \in \mathbb{R}^N : 1/2 < |x| < 1 \} \) and for \( \rho > 0 \) set \( D_\rho = \rho D = \{ \rho x : x \in D \} \). By a suitable choice of \( \rho_0 > 0 \) it follows that the domain
\[
\Omega_\rho = \rho e_N + D_\rho = \{ x : |x - \rho e_N| < \rho \}
\]
lies in \( \Omega \) for all \( 0 < \rho < \rho_0 \).

Following [3] we introduce the auxiliary function \( v \in C^{1,\alpha}(\bar{\Omega}_\rho) \) defined as the weak solution to the problem
\[
L v + \sum_{i=1}^{N} b_i \partial_i v + cv = 0 \quad x \in \Omega_\rho
\]
\[
v = 1 \quad x \in \partial \Omega_\rho^-
\]
\[
v = 0 \quad x \in \partial \Omega_\rho^+,
\]
where \( \partial \Omega_\rho^- = \{ x : |x - \rho e_N| = \rho/2 \} \) and \( \partial \Omega_\rho^+ = \{ x : |x - \rho e_N| = \rho \} \). Existence and uniqueness of a positive solution to (2.1) is provided in Lemma 3.1 below.

It is clear that a small \( \varepsilon > 0 \) can be found so that
\[
u = u - u_0 - \varepsilon v \geq 0,
\]
on \( \partial \Omega_\rho \) meanwhile
\[
L(u - u_0 - \varepsilon v) + c(u - u_0 - \varepsilon v) \geq 0,
\]
in the weak sense in \( \Omega_\rho \). The weak maximum principle [5] then implies that \( u \geq u_0 + \varepsilon v \), in \( \Omega_\rho \). In particular,
\[
\frac{\partial u}{\partial \nu}(0) \leq \varepsilon \frac{\partial v}{\partial \nu}(0), \tag{2.2}
\]
for any outward direction \( \nu \) to \( \Omega_\rho \) at \( x = 0 \). It follows from Lemma 3.4 below that
\[
\frac{\partial v}{\partial \nu}(0) \to -\infty \tag{2.3}
\]
as \( \rho \to 0+ \). An even more precise account on the asymptotic behavior of such derivative as \( \rho \to 0+ \) is given in Lemma 3.4. It is clear that (2.2) and (2.3) imply the desired conclusion (1.5).

**Remark 2.1.** The following strong comparison principle is stated in [6, Proposition 2.2]. Functions \( u, v \in C^1(\overline{\Omega}), u = v = 0 \) on \( \partial \Omega \), solve \(-\Delta_p u = f \) and \(-\Delta_p v = g \) in a smooth bounded domain \( \Omega \subset \mathbb{R}^N \). It is assumed that \( f, g \in L^\infty(\Omega), 0 \leq f \leq g \) while the set \( \{ x \in \Omega : f(x) = g(x) \text{ a. e.} \} \) has an empty interior. Then \( v(x) > u(x) \) for all \( x \in \Omega \) together with

\[
\frac{\partial v}{\partial n} < \frac{\partial u}{\partial n},
\]

at every point in \( \partial \Omega \).

As for its proof, by the contradiction argument employed in [6] it follows that \( v > u \) in \( \Omega \). This is achieved by using weak comparison [12] and the strong maximum principle [13], the latter implying that \( \frac{\partial u}{\partial n} < 0 \) on \( \partial \Omega \). Authors in [6] then obtain (2.4) from the strict inequality between \( u \) and \( v \) in \( \Omega \).

However, we think that to attain (2.4) a more work is required and propose the following argument. Fix \( x_0 \in \partial \Omega \) and assume that contrary to (2.4) the equality

\[
\frac{\partial v}{\partial n}(x_0) = \frac{\partial u}{\partial n}(x_0)
\]

holds. Then there exists a small ball \( B \), centered at \( x_0 \), such that

\[
\min\{|\nabla u(x)|, |\nabla v(x)|\} \geq k > 0
\]

in \( U := B \cap \Omega \). Thus, the difference \( w = v - u \) solves in \( U \) an elliptic equation of the form (1.1) with the uniform elliptic matrix

\[
A(x) = \int_0^1 |\nabla w_t|^{p-2} \left( I + (p-2) \frac{\nabla w_t}{|\nabla w_t|} \otimes \frac{\nabla w_t}{|\nabla w_t|} \right) \, dt,
\]

where \( w_t = (1-t)u + tv, I \) is the identity matrix and for \( \xi \in \mathbb{R}^N, (\xi \otimes \xi)_{ij} = \xi_i \xi_j \), \( 1 \leq i, j \leq N \). Since \( \frac{\partial u}{\partial n}(x_0) \neq 0 \) this implies, by reducing \( B \) if necessary, that \( 0 \notin [\nabla u(x), \nabla v(x)] \) for all \( x \in \overline{U} \). Equivalently, that \( |\nabla w_t(x)| > 0 \) for all \( x \in \overline{U}, t \in [0, 1] \). Taking into account that \( u, v \in C^{1,\alpha}(\overline{\Omega}) \) for some \( 0 < \alpha < 1 \) then the coefficients \( a_{ij} \) of matrix \( A \) in (2.6) belong to \( C^{\alpha}(\overline{U}) \). In this respect it should be remarked that \( \nabla u(x) \neq 0 \) and \( \nabla v(x) \neq 0 \) in \( \overline{U} \) are not enough to ensure us that \( a_{ij} \in C^{\alpha}(\overline{U}) \). Finally, Theorem 1.1 can now be used to conclude that (2.5) is not possible. Hence, (2.4) holds at \( x_0 \).

3. Auxiliary results

**Lemma 3.1.** Problem (2.1) admits a unique positive solution \( v \in C^{1,\alpha}(\overline{\Omega}_\rho) \).

**Proof.** Existence of a unique weak solution \( v \in H^1(\Omega_\rho) \) to (2.1) is standard [5, Theorem 8.3], being the uniqueness consequence of the weak maximum principle. Just this result implies that \( 0 \leq v \leq 1 \text{ a. e. in } \Omega \). Since \( v \in L^\infty(\Omega) \), classical results in [5, Section 8.11] imply that \( v \in C^{\beta}(\overline{\Omega}_\rho) \) for some \( 0 < \beta < 1 \). Furthermore, strong maximum principle [5, Theorem 8.19] ensures us that \( v(x) > 0 \) for all \( x \in \Omega_\rho \). Also the results in [5, Section 8.11] permit us concluding that \( v \in C^{1,\alpha}(\overline{\Omega}_\rho) \). □
Remark 3.2. When $b_i \equiv 0$, $1 \leq i \leq N$, in (2.1) existence of a weak solution can be directly obtained by a variational argument. In fact, the functional

$$J(u) = \frac{1}{2} \int_{\Omega_{\rho}} \left\{ \sum_{i,j=1}^{N} a_{ij} \partial_i u \partial_j u + cu^2 \right\},$$

is coercive in $M = \{ u \in H^1(\Omega_{\rho}) : u = \varphi \text{ on } \partial \Omega_{\rho} \}$, where $\varphi$ is the characteristic function of $\partial \Omega_{\rho}$ in $\partial \Omega_{\rho}$. It therefore admits a global minimizer $u \in H^1(\Omega_{\rho})$ in $M$. Moreover, such minimizer is unique due to the convexity of $J$ ($c \geq 0$).

Consider now the elliptic operator

$$\mathcal{L}u = -\sum_{i,j=1}^{N} \tilde{a}_{ij} \partial_{ij} u,$$

where the coefficients $\tilde{a}_{ij}$ are constant and the matrix $\tilde{A} = (\tilde{a}_{ij})$ is symmetric and positive definite with eigenvalues $0 < \tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_N$.

Let $D$ be the unitary annulus introduced above and $D_{\rho}$ the corresponding annulus with exterior radius $\rho$. Set $G_{\rho}(x,y)$ the Green function associated to $\mathcal{L}$, under homogeneous Dirichlet conditions in $D_{\rho}$ (see [2]). Namely, the unique function $G_{\rho} \in C^2(\overline{D_{\rho}} \times \overline{D_{\rho}} \setminus \Delta)$, $\Delta = \{(x,x) : x \in \overline{D_{\rho}} \}$, such that the classical solution $u \in C^2(D_{\rho}) \cap C(\overline{D_{\rho}})$ to the problem

$$\mathcal{L}u = f \quad x \in D_{\rho}$$

$$u = 0 \quad x \in \partial D_{\rho},$$

with $f \in C(D_{\rho}) \cap L^1(D_{\rho})$, provided that it exists, can be represented in the form

$$u(x) = \int_{D_{\rho}} G_{\rho}(x,y)f(y) \, dy. \quad (3.2)$$

The next result provides us with the key estimates on the derivatives of $G_{\rho}$.

**Lemma 3.3.** There exist positive constants $C_1, C_2$ such that

$$|\partial_x G_{\rho}(x,y)| \leq \frac{C_1}{|x-y|^{N-1}} \quad 1 \leq i \leq N, \quad (3.3)$$

$$|\partial_x \partial_y G_{\rho}(x,y)| \leq \frac{C_2}{|x-y|^N} \quad 1 \leq i, j \leq N, \quad (3.4)$$

for all $x,y \in D_{\rho}$, $x \neq y$. Moreover, constants $C_1$ and $C_2$ can be estimated as follows:

$$C_1 \leq K_1 \left( \frac{\lambda_N}{\lambda_1} \right)^{N-1} \frac{1}{\lambda_1}, \quad C_2 \leq K_2 \left( \frac{\lambda_N}{\lambda_1} \right)^N \frac{1}{\lambda_1}, \quad (3.5)$$

where the positive constants $K_1$ and $K_2$ do not depend on $\rho$.

**Proof.** There exists a linear isomorphism $y = Tx$ which maps $D_{\rho}$ into the ellipsoidal cavity $\mathcal{D}_{\rho} = \{ py : y \in D \}$ with

$$D = \{ y \in \mathbb{R}^N : \frac{1}{4} < \sum_{i=1}^{N} \frac{y_i^2}{a_i^2} < 1 \},$$
and where the reference semiaxis \(a_i\) are given by
\[
a_i = \frac{1}{\sqrt{\lambda_i}}, \quad i = 1, \ldots, N.
\]

Moreover, \(T\) transforms problem (3.1) into
\[
-\Delta v = g \quad y \in D_\rho
\]
\[
v = 0 \quad y \in \partial D_\rho,
\]
where \(v(y) = u(T^{-1}y),\ g(y) = f(T^{-1}y)\). Let \(\tilde{G}_\rho = \tilde{G}_\rho(y, \eta)\) be the Green function associated to \(-\Delta\) under homogeneous Dirichlet conditions in \(D_\rho\). A direct computation shows that
\[
G_\rho(x, \xi) = \{\det T\} \tilde{G}_\rho(Tx, T\xi),
\]
for all \(x, \xi \in D_\rho\), \(x \neq \xi\), where \(\det T = a_1 \cdots a_N\). A further scaling argument permits us writing
\[
\tilde{G}_\rho(y, \eta) = \rho^{2-N} \{\det T\} G \left( \frac{y}{\rho}, \frac{\eta}{\rho} \right), \quad y, \eta \in D_\rho, \ y \neq \eta,
\]
\(G = G(z, \zeta)\) being the Green function for \(-\Delta\), constrained with homogeneous Dirichlet conditions in \(D\). Therefore,
\[
G_\rho(x, \xi) = \rho^{2-N} \{\det T\} G \left( \frac{Tx}{\rho}, \frac{T\xi}{\rho} \right) \quad x, \xi \in D_\rho \quad x \neq \xi.
\]
Now, the estimates in [14] allow us assert the existence of a positive constant \(M\) such that
\[
|\partial_{x_i} G(x, y)| \leq \frac{M}{|x-y|^{N-1}} \quad 1 \leq i \leq N,
\]
\[
|\partial_{x_i} \partial_{y_j} G(x, y)| \leq \frac{M}{|x-y|^N} \quad 1 \leq i, j \leq N,
\]
for all \(x, y \in D, \ x \neq y\). Next we observe that isomorphism \(T\) can be chosen of the form
\[
T = \text{diag}(a_1, \ldots, a_N) \ L,
\]
where \(L\) is an orthogonal transformation. Thus,
\[
\partial_{x_i} G_\rho(x, y) = \rho^{1-N} \sum_{k=1}^{N} \partial_{x_k} G \left( \frac{Tx}{\rho}, \frac{Ty}{\rho} \right) \partial_{x_i} ((Tx)_k),
\]
where \((Tx)_k = a_k \sum L_{xs} x_s\). Then, the estimate
\[
\sum_{k=1}^{N} |\partial_{x_i} ((Tx)_k)| \leq \sqrt{Na_1}
\]
follows easily. In addition,
\[
|Tx| = |\text{diag}(a_1, \ldots, a_N)Lx| \geq a_1|x|,
\]
for all \(x \in \mathbb{R}^N\). By (3.7) with the last estimates, the first inequality in (3.5) is obtained with the choice
\[
K_1 = M\sqrt{N}.
\]
By proceeding in the same way, the second inequality in (3.5) holds for \(K_2 = MN\). \(\square\)
Lemma 3.4. Let $v \in C^{1, \alpha}(\Omega_\rho)$ be the positive solution of the problem (2.1). Then,
\[
\partial v \partial \nu(0) \sim \frac{C^*_N}{\rho} \langle \nu, e_N \rangle \quad \text{as} \quad \rho \to 0^+,
\]
where $\nu \in \mathbb{R}^N$ is any unitary vector and
\[
C^*_N = \frac{N - 2}{2N - 2} - 1.
\]

Remark 3.5. Observe that exterior directions $\nu$ to $\Omega_\rho$ at $x = 0$ are characterized by $\langle \nu, e_N \rangle < 0$.

Proof of Lemma 3.4. To prove (3.8) we follow the argument in [3] and introduce $u = \psi$, the solution of the problem
\[
\begin{align*}
\Delta u &= 0 \quad x \in \Omega_\rho \\
u &= 1 \quad x \in \partial \Omega^-_\rho \\
u &= 0 \quad x \in \partial \Omega^+_\rho,
\end{align*}
\]
whose explicit form is
\[
\psi(x) = \left( \frac{1}{|x - \rho e_N|^{N-2}} - \frac{1}{\rho^{N-2}} \right) \frac{\rho^{N-2}}{2N-1}.
\]

We fix now $\bar{x} \in \Omega_\rho$ and define the constant coefficients operator
\[
L_{\bar{x}}u := -\sum_{i,j=1}^N \bar{a}_{ij} \partial_{ij} u,
\]
with $\bar{a}_{ij} = a_{ij}(\bar{x})$. By noticing that $w(x) := v(x) - \psi(x)$ vanishes at the boundary $\partial \Omega_\rho$ of $\Omega_\rho$, $w$ can be represented as
\[
w(x) = \int_{\Omega_\rho} G_\rho(x, y) L_{\bar{x}} w(y) \, dy,
\]
where $G_\rho$ stands for the Green function of the operator $L_{\bar{x}}$ in $\Omega_\rho$, subject to homogeneous Dirichlet conditions on $\partial \Omega_\rho$ (see Lemma 3.3). We are employing (3.9) to analyze $\nabla w$ near zero when $\rho$ becomes small.

Observe that,
\[
w(x) = \int_{\Omega_\rho} G_\rho(x, y) (L_{\bar{x}} v(y) - L v(y)) \, dy
\]
\[
- \int_{\Omega_\rho} G_\rho(x, y) (L_{\bar{x}} \psi(y) - L_0 \psi(y)) \, dy
\]
\[
- \int_{\Omega_\rho} G_\rho(x, y) (b(y) \nabla v(y) + c(y) v(y)) \, dy
\]
\[
=: w_1(x) + w_2(x) + w_3(x),
\]
x $\in \Omega_\rho$, with $b = (b_i)$ and where $L_0$ is the constant coefficients operator resulting from fixing $x = 0$ in the functions $a_{ij}(x)$. Notice that $L_0 = -\Delta$ and so $L_0 \psi = 0$.

On the other hand, $w_1(x)$
\[
w_1(x) = \sum_{i,j=1}^N \int_{\Omega_\rho} \partial_{i,j} G_\rho(x, y) (a_{ij}(y) - a_{ij}(\bar{x})) \partial_j v(y) \, dy.
\]
Hence,
\[ \partial_{x_s} w_1(\bar{x}) = \sum_{i,j=1}^{N} \partial_{x_s} \partial_{y_i} G_{\rho}(\bar{x}, y)(a_{ij}(y) - a_{ij}(\bar{x})) \partial_{j} v(y) \, dy. \]  
(3.10)

By estimate (3.4) in Lemma 3.3,
\[ |\partial_{x_s} w_1(\bar{x})| \leq N \sum_{i,j=1}^{N} C_2[a_{ij}]_\alpha \|\nabla v\|_{\infty, \Omega} \int_{\Omega_{\rho}} \frac{1}{|y - \bar{x}|^{N-\alpha}} \, dy, \]
where
\[ [a_{ij}]_\alpha = \sup_{x,y \in \Omega, x \neq y} \frac{|a_{ij}(x) - a_{ij}(y)|}{|x - y|^\alpha}. \]

After estimating the integral, (3.10) implies that
\[ |\nabla w_1(\bar{x})| \leq C \|\nabla v\|_{\infty, \Omega} \rho \bar{x} \in \Omega_{\rho}, \] 
(3.11)
for a certain positive constant $C$ which does not depend on $\rho$. Label $C$ will be employed in the sequel to designate positive constants which no depend on $\rho$, and whose precise value is irrelevant for the discourse.

As for the gradient of $w_2$,
\[ \partial_{x_s} w_2(\bar{x}) = \sum_{i,j=1}^{N} \partial_{x_s} G_{\rho}(\bar{x}, y)(a_{ij}(0) - a_{ij}(\bar{x})) \partial_{j} \psi(y) \, dy. \]

Since $|\partial_{ij} \psi(y)| \leq C \rho^{-2}$, using estimate (3.3) we find that
\[ |\partial_{x_s} w_2(\bar{x})| \leq \sum_{i,j=1}^{N} C[a_{ij}]_\alpha \rho^{\alpha-2} \int_{\Omega_{\rho}} \frac{1}{|y - \bar{x}|^{N-1}} \, dy. \]

By estimating the integral in terms of $\rho$ we obtain
\[ |\partial_{x_s} w_2(\bar{x})| \leq C \rho^{\alpha-1} \quad \bar{x} \in \Omega_{\rho}. \]  
(3.12)

On the other hand, taking into account that $v(0) = 0$, we conclude that
\[ |\partial_{x_s} w_3(\bar{x})| \leq C_1 \|c\|_{\infty, \Omega} \|\nabla v\|_{\infty, \Omega} \int_{\Omega_{\rho}} \frac{1}{|y - \bar{x}|^{N-1}} \, dy, \]
and so,
\[ |\partial_{x_s} w_3(\bar{x})| \leq C \|\nabla v\|_{\infty, \Omega} \rho^2 \quad \bar{x} \in \Omega_{\rho}. \]  
(3.13)

From (3.11), (3.12) and (3.13) the estimate
\[ \|\nabla w\|_{\infty, \Omega_{\rho}} \leq C \|\nabla v\|_{\infty, \Omega_{\rho}} \rho^{\alpha} + C \rho^{\alpha-1} \]  
(3.14)
holds.

Now, $\|\nabla v\|_{\infty, \Omega_{\rho}}$ can be estimated in terms of $\rho$. In fact,
\[ |\nabla v(x)| \leq |\nabla w(x)| + |\nabla \psi(x)| \leq \|\nabla w\|_{\infty, \Omega_{\rho}} + C \rho^{-1} \quad x \in \Omega_{\rho}. \]

Hence,
\[ \|\nabla v\|_{\infty, \Omega_{\rho}} \leq C \rho^{-1}, \]
which, together with (3.14), imply that
\[ \|\nabla w\|_{\infty, \Omega_{\rho}} \leq C \rho^{\alpha-1}. \]  
(3.15)
Finally,
\[ |\frac{\partial v}{\partial \nu}(0) - \frac{\partial \psi}{\partial \nu}(0)| = |\frac{\partial v}{\partial \nu}(0) - \frac{C_N^*}{\rho} \langle \nu, e_N \rangle| \leq \|\nabla w\|_{\infty, \Omega_\rho} \leq C \rho^{\alpha - 1}.\]

Thus,
\[ \frac{C_N^*}{\rho} \left( \langle \nu, e_N \rangle - \frac{C}{C_N^*} \rho^\alpha \right) \leq \partial_\nu v(0) \leq \frac{C_N^*}{\rho} \left( \langle \nu, e_N \rangle + \frac{C}{C_N^*} \rho^\alpha \right),\]

for \( \rho \) small. Asymptotic estimate (3.8) immediately follows from these inequalities. \( \square \)

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**References**


