Tema 2

- Dinámica de ecuaciones escalares: estabilidad y bifurcación.
- Un modelo para las explosiones demográficas de la oruga del abeto.
- Histéresis y catástrofe de cúspide.
- Gestión de recursos renovables.

Equilibrios, estabilidad, comportamiento asintótico.

Para muchas de las magnitudes que estudiaremos (u = nº de individuos, concentración química, preferiblemente) tendremos una ley de evolución:

$$\frac{du}{dt} = f(u) \qquad (1)$$

- $f(u)$ = ley de crecimiento
  = velocidad / tasa de reacción

- ¿Cuál es la predicción a largo plazo?

- Para (1) las soluciones de equilibrio:

$$u(t) = u^* = cte,$$

que han de cumplir:

$$f(u^*) = 0$$

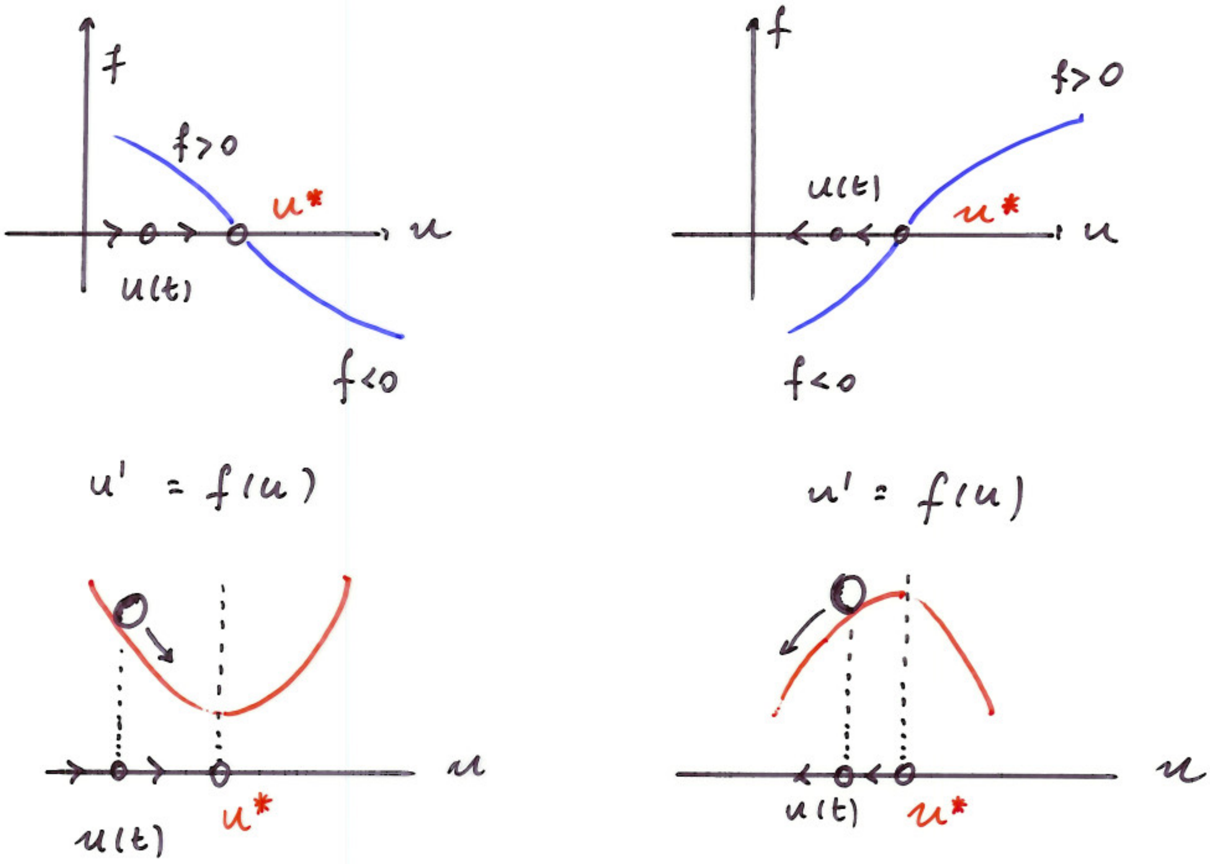
- $f'(u^*) < 0$; $u = u^*$ estable $\Rightarrow$

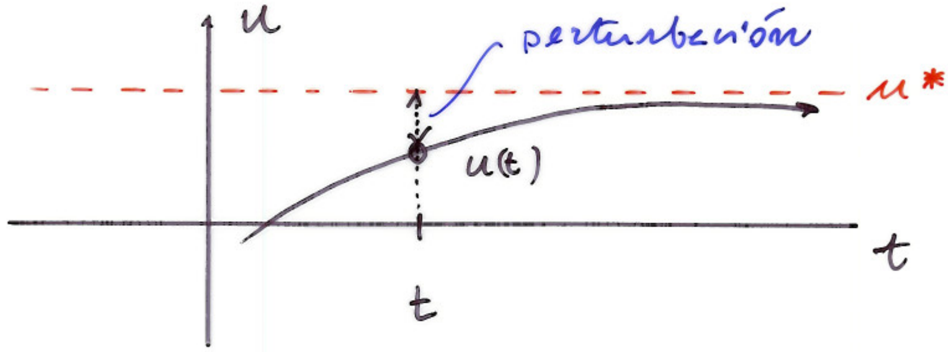
$$u(t) \longrightarrow u^* \quad \text{si} \quad t \to \infty$$

y $u(t_0)$ situado adecuadamente con respecto a u^*.

- $f'(u^*) > 0$; $u = u^*$ inestable $\Rightarrow$

$u(t)$ se aleja de u^*.





Ecuación de la perturbación:

$$u' = f(u)$$

$$0 = (u^*)' = f(u^*)$$

$$V(t) = u(t) - u^* = \text{perturbación}$$

$$v' = f(u) = f(u^* + v)$$

$$= f'(u^*)\, v + \cancel{o(v)}$$

$$v' = f'(u^*)\, v.$$

Conclusión:

$$v(t) = e^{f'(u^*) t}\, v_0$$

$[f'(u^*)]^{-1}$ = tiempo característico de decaimiento al equilibrio

$f'(u^*)$ = exponente de Lyapunov.

Oruga del Abeto
(Spruce budworm)

Abeto: spruce, fir.

- Los bosques boreales, especialmente Canadá sustentan pocas especies de insectos: sin depredadores invertebrados.

- Con cierta periodicidad sufren explosiones en la población ("outbreaks") en forma de plagas ("pests").

- Este es el caso de la oruga del abeto. La explosión demográfica se automantiene a lo largo de 4 años en las que desforestan bastas áreas de follaje
  - plaga 4 años
  - decae la población de abetos en beneficio de los abedules
  - abedules y abetos compiten por los nutrientes
  - Los abetos predominan y la población vuelve a regenerarse al cabo de un periodo de 50 a 100 años.

Un modelo de crecimiento de la población de orugas del abeto(*):

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - p(N)$$

r_B = tasa de crecimiento intrínseca

K_B = capacidad de soporte ~ densidad de masa arbórea / follaje.

$p(N)$: término de depredación por pájaros

* $p(N)$ se satura cuando $N \to \infty$

* $p(N)$ es despreciable cuando $N \ll 1$: los pájaros no "distinguen" la población y van a depredar a otras áreas

* $p(N)$ se dispara o activa en un valor crítico N_c

(*) Ludwig / Jones / Holling J. Animal Ecology 47 (1978), 315-332

Término de Depredación:

$$p(N) = \frac{BN^2}{A^2 + N^2}$$

Variables adimensionales:

$$u = \frac{N}{A}$$

$$r = \frac{A r_B}{B}$$

$$q = \frac{K_B}{A}$$

$$\tau = \frac{Bt}{A}$$

Ecuación adimensional

$$\frac{du}{d\tau} = r u \left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2}$$

. Parámetros : q, r

. Curva de bifurcación:

$$q = \frac{2u^3}{u^2 - 1} \qquad r = \frac{2u^3}{(1 + u^2)^2} \qquad u > 1$$

The steady states are solutions of

$$f(u;r,q) = 0 \quad \Rightarrow \quad ru\left(1 - \frac{u}{q}\right) = \frac{u^2}{1+u^2}\,. \tag{1.9}$$

Clearly $u = 0$ is one solution with other solutions, if they exist, satisfying

$$r\left(1 - \frac{u}{q}\right) = \frac{u}{1+u^2}\,. \tag{1.10}$$

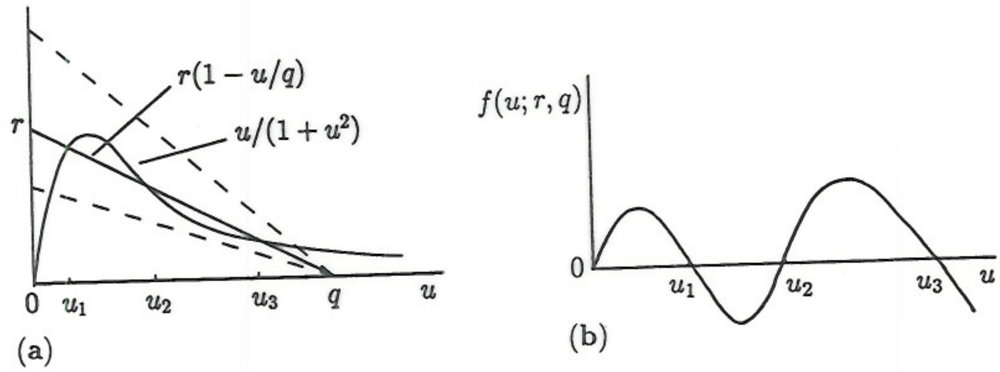


Fig. 1.4a,b. Equilibrium states for the spruce budworm population model (1.8). The positive equilibria are given by the intersections of the straight line $r(1-u/q)$ and $u/(1+u^2)$. With the solid straight line in (a) there are 3 steady states with $f(u;r,q)$ typically as in (b).

Although we know the analytical solutions of a cubic (Appendix 2), they are often clumsy to use because of their algebraic complexity; this is one of these cases. It is convenient here to determine the existence of solutions of (1.10) graphically as shown in Fig. 1.4 (a). We have plotted the straight line, the left of (1.10), and the function on the right of (1.10); the intersections give the solutions. The actual expressions are not important here. What is important, however, is the existence of one, three, or again, one solution as r increases for a fixed q, as in Fig. 1.4 (a), or as also happens for a fixed r and a varying q. When r is in the appropriate range, which depends on q, there are three equilibria with a typical corresponding $f(u;r,q)$ as shown in Fig. 1.4 (b). The nondimensional groupings which leave the two parameters appearing only in the straight line part of Fig. 1.4 is particularly helpful and was the motivation for the form introduced in (1.7). By inspection $u = 0$, $u = u_2$ are linearly unstable, since $\partial f/\partial u > 0$ at $u = 0, u_2$, while u_1 and u_3 are stable steady states, since at these $\partial f/\partial u < 0$. There is a domain in the r,q parameter space where three roots of (1.10) exist. This is shown in Fig. 1.5: the analytical derivation of the boundary curves is left as an exercise (Exercise 1).

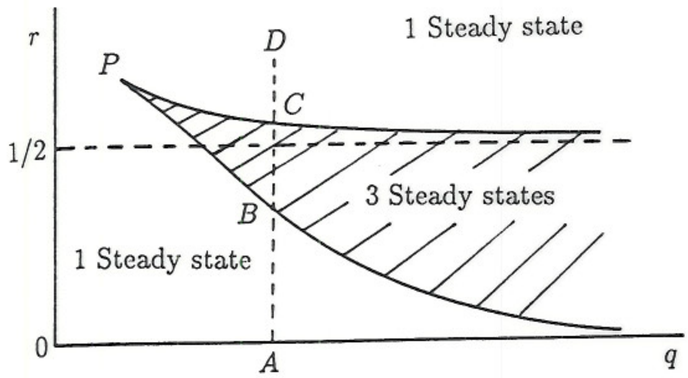


Fig. 1.5. Parameter domain for the number of positive steady states for the budworm model (1.8). The boundary curves are given parametrically (see Exercise 1) by $r(a) = 2a^3/(a^2+1)^2$, $q(a) = 2a^3/(a^2-1)$ for $a \geq \sqrt{3}$, the value giving the cusp point P.

This model exhibits a *hysteresis effect.* Suppose we have a fixed q, say, and r increases from zero along the path $ABCD$ in Fig. 1.5. Then, referring also to Fig. 1.4 (a), we see that if $u_1 = 0$ at $r = 0$ the u_1-equilibrium simply increases monotonically with r until C in Fig. 1.5 is reached. For a larger r this steady state disappears and the equilibrium value jumps to u_3. If we now reduce r again the equilibrium state is the u_3 one and it remains so until r reaches the lower critical value, where there is again only one steady state, at which point there is a jump from the u_3 to the u_1 state. In other words as r increases along $ABCD$ there is a discontinuous jump up at C while as r decreases from D to A there is a discontinuous jump down at B. This is an example of a *cusp catastrophe* which is illustrated schematically in Fig. 1.6 where the letters A, B, C and D correspond to those in Fig. 1.5. Note that Fig. 1.5 is the projection of the surface onto the r, q plane with the shaded region corresponding to the fold.

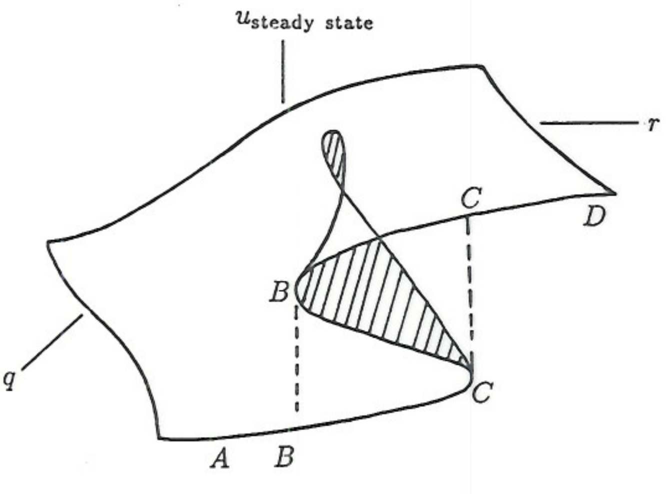


Fig. 1.6. Cusp catastrophe for the equilibria states in the $(u_{\text{steady state}}, r, q)$ parameter space. As r increases from A the path is $ABCCD$ while as r decreases from D the path is $DCBBA$. The projection of this surface onto the r, q plane is given in Fig. 1.5. Three equilibria exist where the fold is.

The parameters from field observation are such that there are three possible steady states for the population. The smaller steady state u_1 is the *refuge* equilibrium while u_3 is the *outbreak* equilibrium. From a pest control point of view, plaga what should be done to try to keep the population at a refuge state rather than allow it to reach an outbreak situation? Here we must relate the real parameters to the dimensionless ones, using (1.7). For example, if the foliage was sprayed to

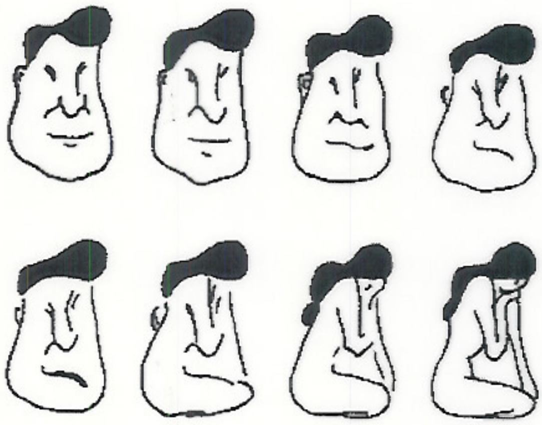


Figure 1.8. Series of pictures exhibiting abrupt (catastrophic) visual change during the variation from a man's face to a sitting woman. (After Fisher 1967)

The picture numbered 1 in the figure is clearly a man's face while picture 8 is clearly a sitting woman. An experiment which demonstrates the sudden jump from seeing the face of a man to the picture of the woman is described by Zeeman (1982). The author carried out a similar experiment with a group of 57 students, none of whom had seen the series before nor knew of such an example of sudden change in visual perception. The experiment consisted of showing the series three times starting with the man's face, picture 1, going up to the woman, picture 8, then reversing the sequence down to 1 and again in ascending order; that is, the figures were shown in the order 123456787654321234 5678. The students were told to write down the numbers where they noticed a major change in their perception; the results are presented in Table 1.2.

The predictions were that during the first run through the series, that is, from 1 to 8, the perception of most of the audience would be locked into the figure of the face until it became obvious that the picture was in fact a woman at which stage there would be sudden jump in perception. As the pictures were shown in the reverse order the audience was now aware of the two possibilities and so could make a more balanced judgement as to what a specific picture represented. The perception change would therefore more likely occur nearer the middle, around 5 and 4. During the final run through the series the change would again occur near the middle.

The results of Table 1.2 are shown schematically in Figure 1.9 where the vertical axis is perception, p, and the horizontal axis, the stimulus, the picture number. The

Table 1.2. Numbers for the perceptual catastrophe experiment involving 57 undergraduates, none of whom had seen the series nor had heard of the phenomenon before.

Sequence Direction	Picture Sequence								Mean	Sequence Direction
→	1	2	3	4	5	6	7	8		
number switching	0	0	0	0	5	8	25	19	7.0	
	1	2	3	4	5	6	7	8		←
number switching	0	1	1	17	29	6	3	—	4.8	
→	1	2	3	4	5	6	7	8		
number switching	—	0	3	19	19	12	3	1	4.9	

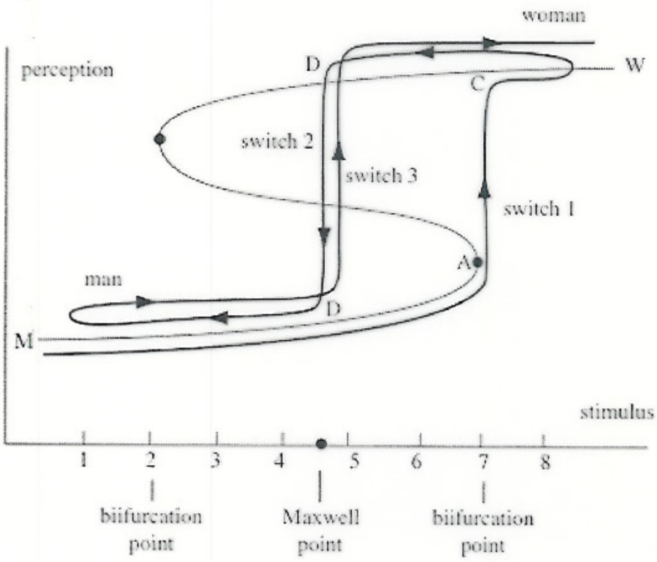


Figure 1.9. Schematic representation of the visual catastrophe based on the data in Table 1.2 on three runs (1234567876543212345678) through the series of pictures in Figure 1.8. The stimulus switches occurred at points denoted by 1, 2 and 3 in that order.

graph of perception versus stimulus is multi-valued in a traditional cusp catastrophe way. Here, over part of the range there are two possible perceptions, a face or a woman. The relation with the example of the budworm population problem is clear. On the first run through the switch was delayed until around picture 7 while on the run through the down sequence it occurred mainly at 5 and again around 5 on the final run through the pictures. There is, however, a fundamental difference between the phenomenon here and the budworm. In the latter there is a definite and reproducible hysteresis while in the former this hysteresis effect occurs only once after which the dynamics is single valued for each stimulus.

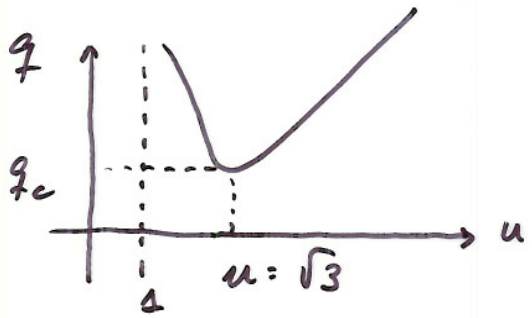
If we had started with picture 8 and again run the series three times the results would be similar except that the jump would have occurred first around 2, as in the figure, with the second and third switch again around 5. The phenomenon of catastrophic change in behavior and perception is widespread and an understanding of the underlying dynamics would clearly be of great help. Several other qualitative examples in psychology are described by Zeeman (1977).

1.3 Delay Models

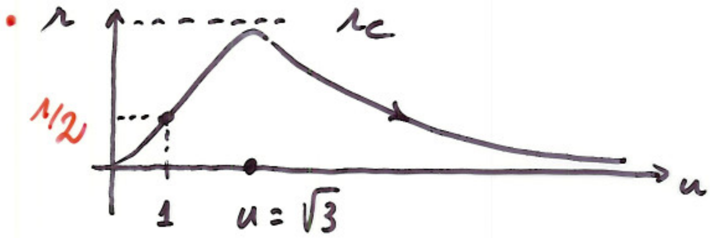
One of the deficiencies of single population models like (1.4) is that the birth rate is considered to act instantaneously whereas there may be a time delay to take account of the time to reach maturity, the finite gestation period and so on. We can incorporate such delays by considering delay differential equation models of the form

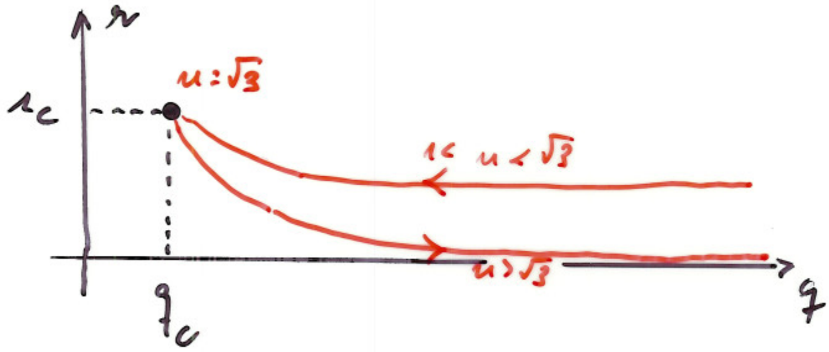
$$\frac{dN(t)}{dt} = f\left(N(t), N(t-T)\right), \qquad (1.11)$$

• Perfil de $q = q(u)$: $q'(u) = \dfrac{2u^2(u^2-3)}{(u^2-1)^2}$



Perfil de $r = r(u)$: $r'(u) = \dfrac{2u^2(3-u^2)}{(1+u^2)^3}$





• Para simulación PHASER:

$$\begin{cases} x' = \dfrac{2t^2(t^2-3)}{(t^2-1)^2} & x(2) = 5'\overline{3} \\[2ex] y' = \dfrac{2t^2(3-t^2)}{(1+t^2)^3} & y(2) = 0'64 \\[2ex] t' = 1 & t(2) = 2 \end{cases}$$

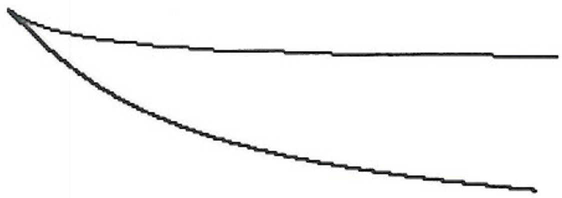
Para más lentitud hacer $t' = \varepsilon$ con ε un parámetro ó paso pequeño.

PHASE PORT:

x min: 0.000000

y min: 0.000000

z min: -10.0000



x max: 25.00000

y max: 1.000000

z max: 10.00000

SETUP:

Equation: spruce Dimension: 3 Algorithm: Euler

Parameters: 1=1.0000

Time Start: 0.000000 Time End: -0.99000

Step Size: -0.01000 Jumps/Plt: 1

3d Projection: X axis: x1 Y axis: x2 Z axis: x3

Map Poincare Plane: 0.000x + 1.000y + 0.000z + 0.000 = 0 P

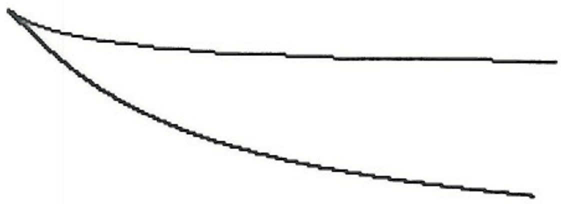
Init Conds: 5.3333 0.6400 2.0000

PHASE PORT:

x min:
0.000000

y min:
0.000000

z min:
-10.0000



x max:
25.00000

y max:
1.000000

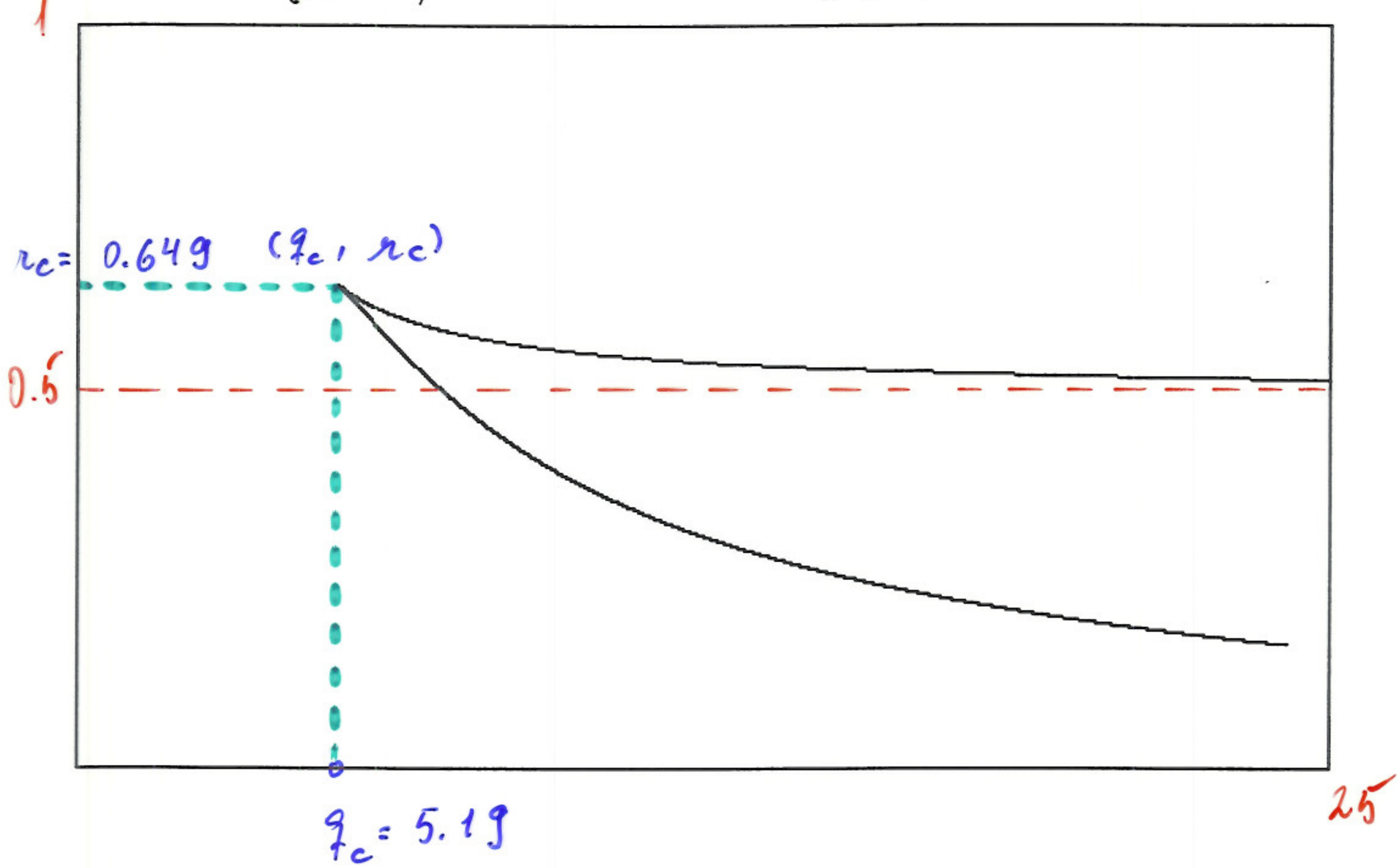
z max:
10.00000

EQUATION:

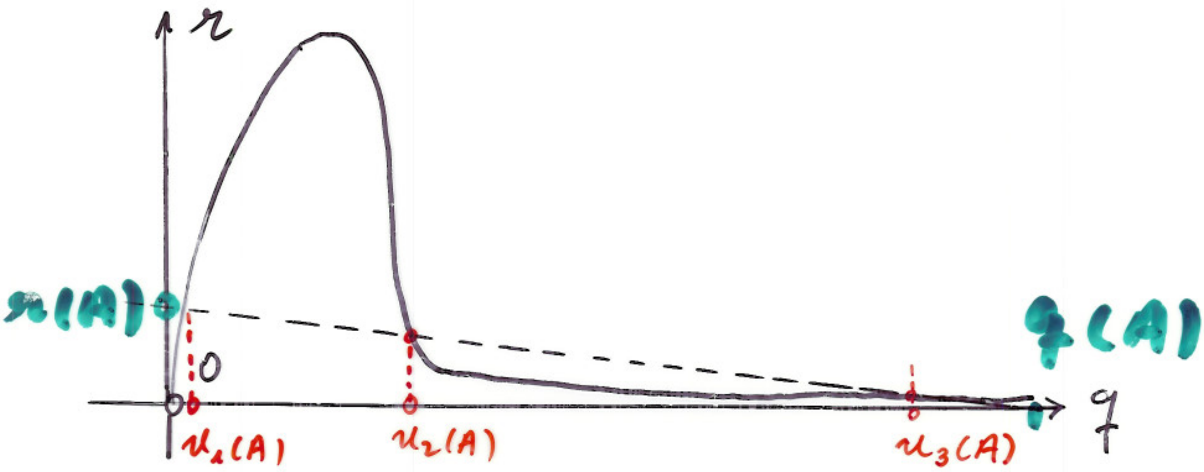
spruce

x1' = 1*(2.000 *x3^2.000 *(x3^2.000 -3.000))/((x3^2.000 -1.000)
^2.000)
x2' = 1*(2.000 *x3^2.000 *(3.000 -x3^2.000))/((1.000 +x3^2.000)
^3.000)
x3' = 1.000

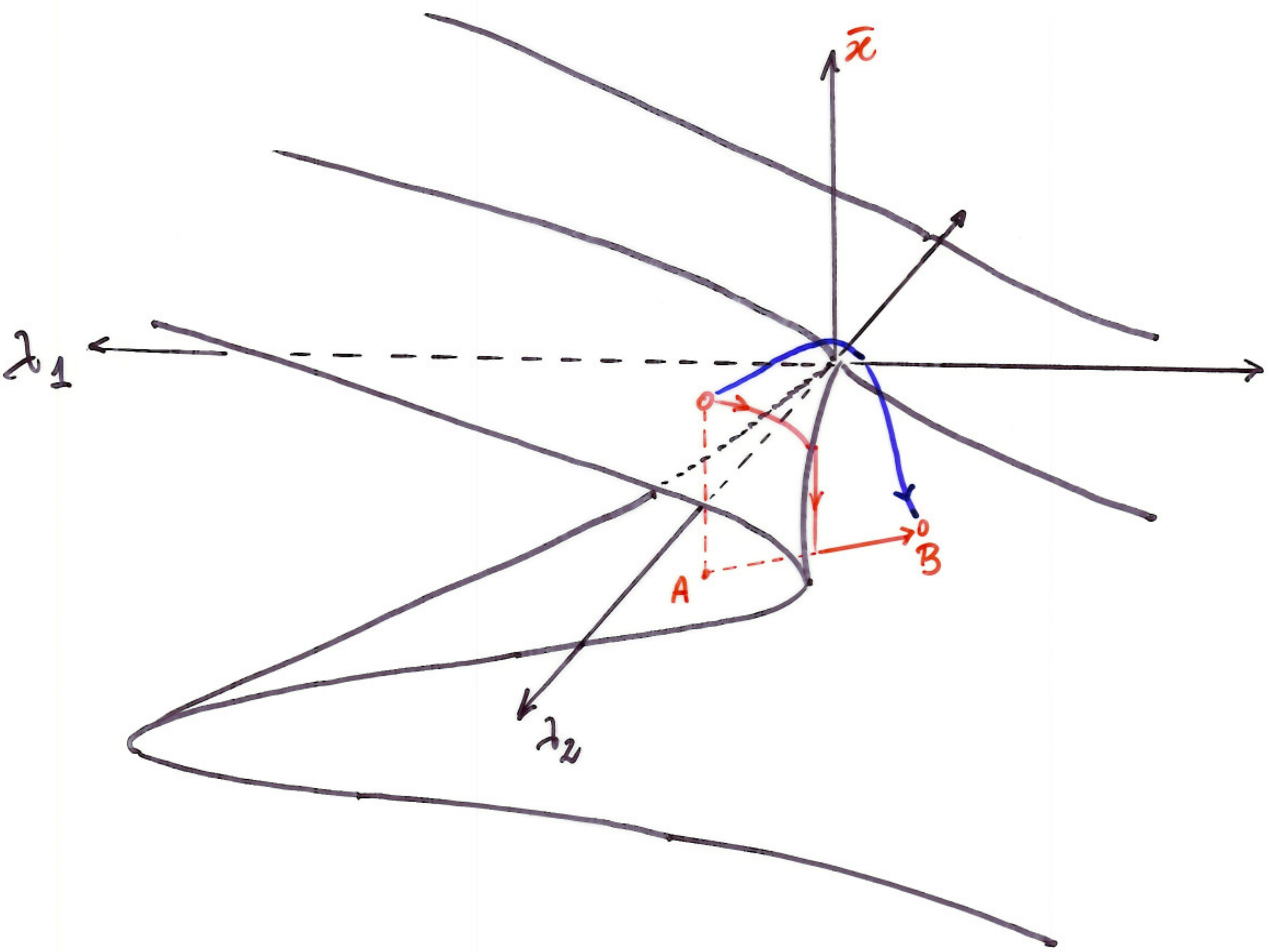
$$r = \frac{2a^3}{(1+a^2)^2} \qquad q = \frac{2a^3}{a^2-1}$$

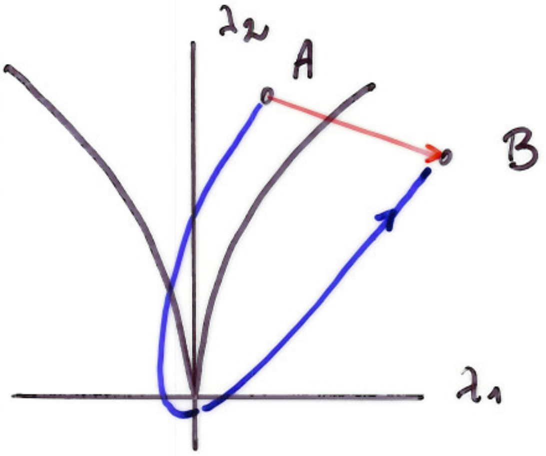


- La presencia del cero intermedio es la posible explicación de las oleadas de población. Si $A \sim 0^+$ en algunos casos las oleadas pueden ser muy violentas.



$$\lambda_1 + \lambda_2 x - x^3 = 0$$

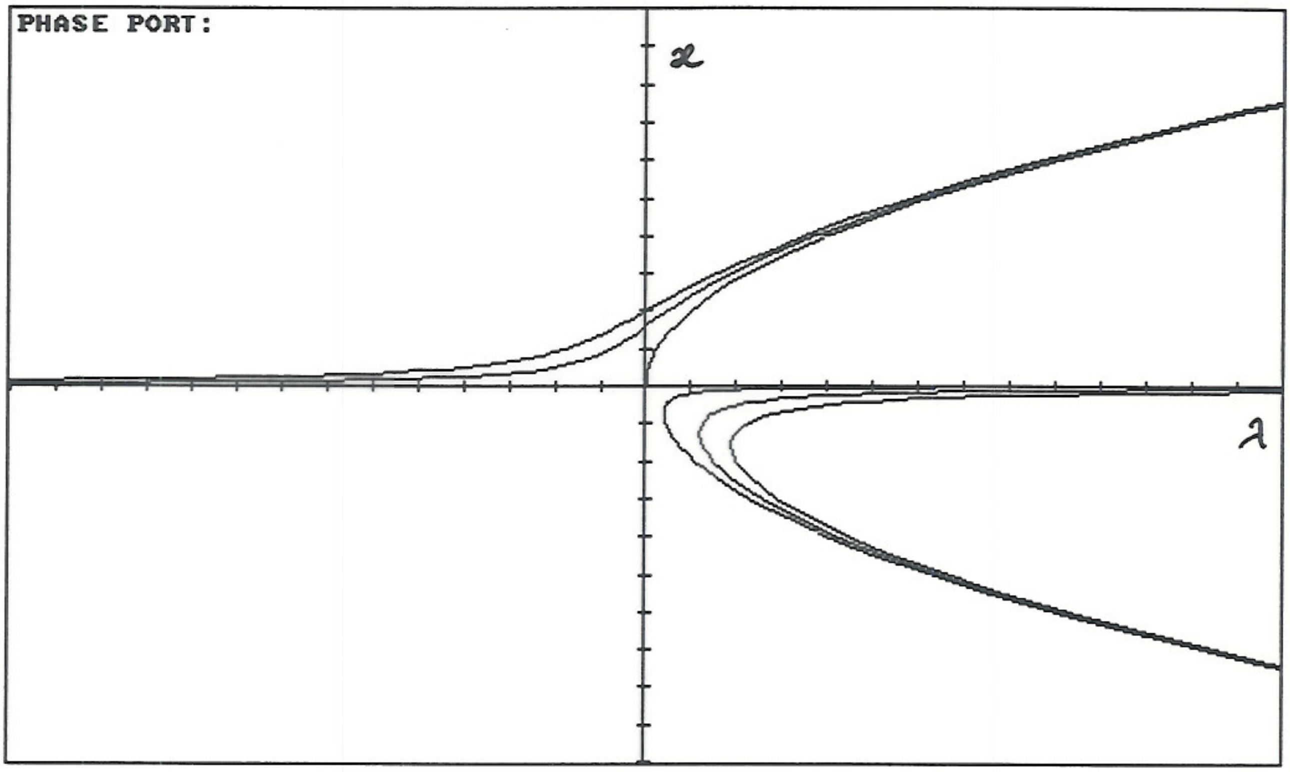




Conjuntos de nivel de

$$\varepsilon + \lambda x - x^3 = 0$$

$\varepsilon = 0.1$, $\varepsilon = 0.5$ $(\varepsilon^{1/3} \sim 0.7937)$, $\varepsilon = 1$



Eje OX	PUNTO DE RETORNO
(0, 0.001)	(1.1, -1)
(0, 0.7937)	(1.5, -1)
(0, 1)	(2, -1)

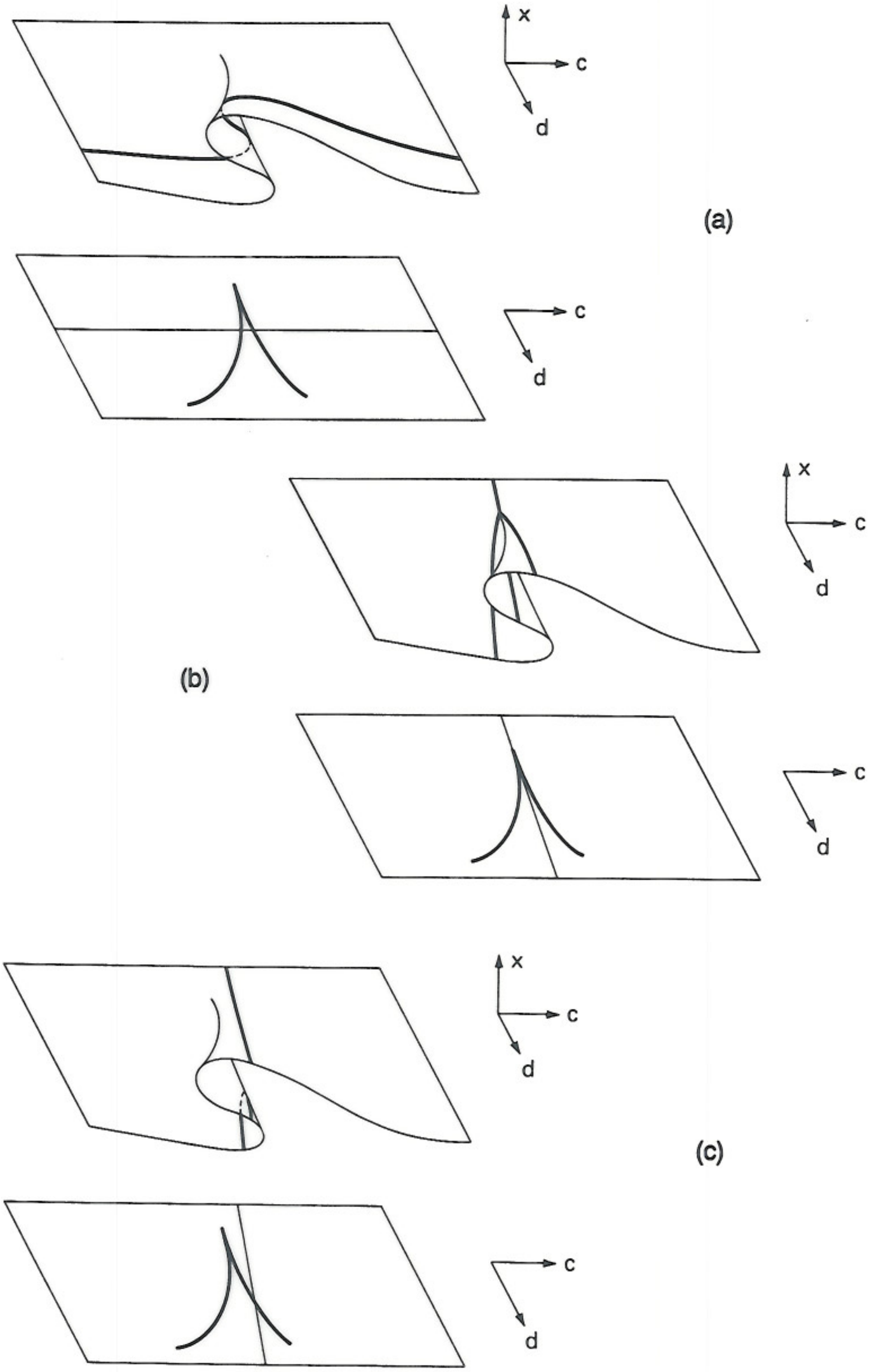


Figure 2.15. *Slices of the bifurcation diagram of* $\dot{x} = c + dx - x^3$: (a) *hysteresis for* $d = 1$, (b) *supercritical pitchfork for* $c = 0$, *and* (c) *supercritical saddle-node for* $c = 1$.

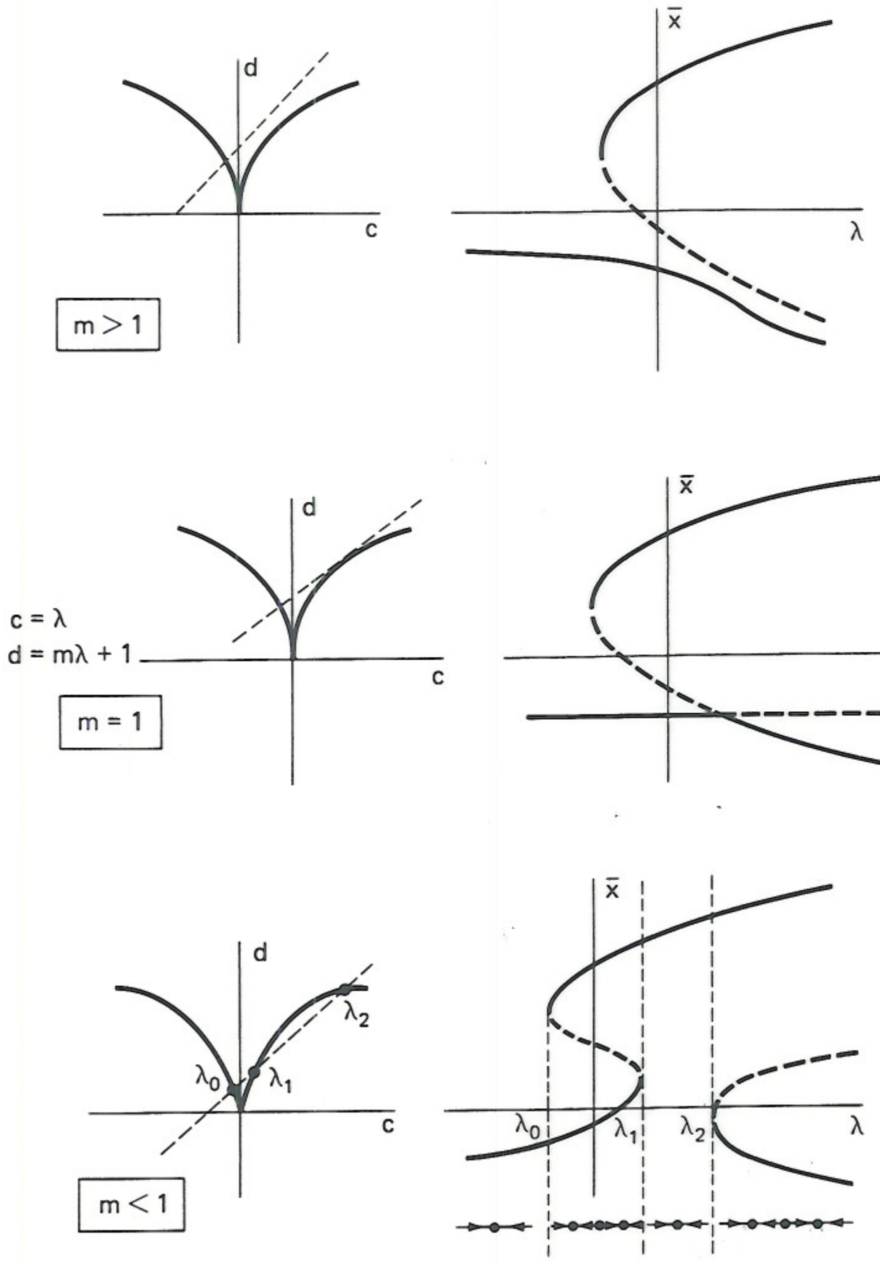


Figure 2.16. *Bifurcation diagrams of* $\dot{x} = \lambda + (m\lambda + 1)x - x^3$ *for* $m > 1$, $m = 1$, *and* $m < 1$.

With these examples at our disposal, we now turn to generalities.

Exercises ♣♡♠◇

2.1. Identify the groups of examples in Exercise 1.5 with the same orbit structure.

2.2. Provide the details of the computation of the bifurcation diagrams in Example 2.7.