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HOPF MAXIMUM PRINCIPLE REVISITED

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Abstract. A weak version of Hopf maximum principle for elliptic equations in divergence form $\sum_{i,j=1}^N \partial_i(a_{ij}(x)\partial_j u) = 0$ with Hölder continuous coefficients a_{ij} was shown in [3], in the two-dimensional case. It was also pointed out that this result could be extended to any dimension. The objective of the present note is to provide a complete proof of this fact, and to cover operators more general than the one studied in [3].

1. Introduction

It is well-known that the Hopf maximum principle (see [5, Lemma 3.4], [10, Theorem II.7] or [11, Theorem 2.8.4] for a classical statement) does not hold for linear elliptic equations in divergence form. More precisely, a function $u \in C^1(\overline{\Omega})$, with $\Omega \subset \mathbb{R}^N$ a smooth domain, is assumed to solve in weak sense the elliptic equation

$$\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = 0 \tag{1.1}$$

in Ω, while $u(x) > u(x_0)$ in an inner tangent ball $B \subset \Omega$, $x_0 \in \partial\Omega \cap \overline{B}$ being the tangency point. Then, a maximum Hopf principle (a "boundary point lemma") holds at x_0 if the strict inequality

$$\frac{\partial u}{\partial n} < 0 \tag{1.2}$$

is satisfied at x_0, n standing for the outward unit normal at that point.

A counterexample to this assertion, even when coefficients a_{ij} in (1.1) are continuous in $\overline{\Omega}$ was given in [5, Problem 3.9] (see also [11, Section 2.7]; and a further example in [9] for the case in which coefficients in (1.1) satisfy $a_{ij} \in L^\infty(\Omega)$). Moreover, as pointed out in [9], a simpler example than the one in [5] can be obtained as follows. Function $u = \Re\frac{z}{\ln z}$, $z = x + iy$, is harmonic and negative in the plane domain Ω enclosed by the C^1 curve $r = \varphi(\theta)$ with $\varphi(\theta) = \exp(-\theta\tan\theta)$ if $|\theta| < \frac{\pi}{2}$, $\varphi(\pm\pi/2) = 0$ ([5], Chapter 3). Outward unit normal at $(0,0) \in \partial\Omega$ is $n = (-1,0)$ while

$$u(0,0) = 0, \quad u_x(0,0) = 0.$$

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Thus (1.2) fails. Since Ω is not a C^2 domain at $(0,0)$ then, after a C^1 rectification of $\partial\Omega$ near $(0,0)$ one finds that u solves an equation (1.1) with respect to new variables (x',y') in $B\cap\{y'>0\}$, with coefficients $a_{ij}\in C(\overline{B}\cap\{y'\geq 0\})$, being B a small ball centered at $(0,0)$. This furnishes us the desired counterexample.

Nevertheless, Hopf maximum principle, when regarded in this weak form, seems to be either not correctly stated (see e. g. [1, Proposition 1.16] where some kind of differentiability assumption on the coefficients seems to be missing) or not properly employed in comparison arguments (proof of [6, Proposition 2.2], Remark 2.1 bellow).

The difficulty in showing a Hopf maximum principle for (1.2) lies, of course, on the lack of differentiability of coefficients a_{ij}. Indeed, a proof in the line of the standard one works provided that the a_{ij} belong to $C^{0,1}(\overline{\Omega})$. That is why it still seems an outstanding result the fact that Hopf principle holds when the a_{ij} are merely Hölder continuous. This was shown in [3, Lemma 7] for (1.1) in the case $N=2$ (a later improved two-dimensional statement appeared in [4]). Moreover, it is asserted in [3, Remark 2 p. 35] that: "The proof of Lemma 7 can be extended to n dimensions for equations of the form (1.1)". Accordingly, the goal of this note is to furnish to the interested reader a detailed proof of such extended version. In addition, the operators we are addressing in the present article are slightly more general than the one announced in [3], meanwhile some of the auxiliary results obtained here seem interesting in its own right (see estimate (3.8) in Lemma 3.4 bellow).

To simplify notation we are using, whenever possible, either ∂_i or ∂_{ij} instead of $\frac{\partial}{\partial x_i}$ or $\frac{\partial^2}{\partial x_i\partial x_j}$, respectively, wherein reference variable x could be replaced by another one, say y depending on the context.

Our main result is stated as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^N$ be a smooth bounded domain, $a_{ij}\in C^\alpha(\overline{\Omega})$ with $a_{ij}(x)=a_{ji}(x)$, $1\leq i,j\leq N$, $x\in\overline{\Omega}$, and*
$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j>0, \tag{1.3}$$
for all $x\in\overline{\Omega}$ and $\xi\in\mathbb{R}^N\setminus\{0\}$. Assume that $u\in C^1(\overline{\Omega})$ solves, in the weak sense,
$$-\sum_{i,j=1}^N \partial_i(a_{ij}(x)\partial_j u)+\sum_{i=1}^N b_i(x)\partial_i u+c(x)u\geq 0, \tag{1.4}$$
in Ω, where $b_i\in L^\infty(\Omega)$ for $1\leq i\leq N$, $c\in L^\infty(\Omega)$ and $c(x)\geq 0$ a. e. in Ω.

Suppose that for $x_0\in\partial\Omega$ there exists a ball $B\subset\Omega$ with $x_0\in\partial B$ where $u=u(x)$ satisfies:
$$u(x)>u(x_0)\quad x\in B.$$
If $u(x_0)\leq 0$ then
$$\frac{\partial u}{\partial\nu}(x_0)<0, \tag{1.5}$$
where ν is any *outward direction, i. e. any unitary vector $\nu\in\mathbb{R}^N$ so that $\langle\nu,n\rangle>0$, n being the outward unitary normal at x_0.*

Remark 1.2. (a) Ball B in the statement is indeed an "inner ball" tangent to $\partial\Omega$ at $x_0\in\partial\Omega$.

(b) No restriction on the sign of c is required in the case where $u(x_0) = 0$. Alternatively, the sign of $u(x_0)$ can be arbitrary provided that $c(x) = 0$ for all $x \in \Omega$.

2. Proof of Theorem 1.1

Define the operator,

$$\mathcal{L}u = -\sum_{i,j=1}^{N} \partial_i (a_{ij}(x)\partial_j u),$$

which is understood to act in weak sense on functions in $C^1(\overline{\Omega})$. Let u be as in the statement of Theorem 1.1 and set $u_0 = u(x_0)$. Then

$$\mathcal{L}(u-u_0) + \sum_{i=1}^{N} b_i \partial_i (u-u_0) + c(u-u_0) \ge -cu_0 \ge 0.$$

By performing a suitable linear transformation of the variable x it can be assumed that $a_{ij}(x_0) = \delta_{ij}$ ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise). After a translation and a rotation it can be also assumed that $x_0 = 0$ meanwhile the outward unit normal n at $x = 0$ is $-e_N$. It should be remarked that after this set of variable changes, outward derivatives of u at x_0 are transformed into outward derivatives of u at 0, with respect to the new variables.

Consider the "unitary" annulus $D = \{x \in \mathbb{R}^N : 1/2 < |x| < 1\}$ and for $\rho > 0$ set $D_\rho = \rho D = \{\rho x : x \in D\}$. By a suitable choice of $\rho_0 > 0$ it follows that the domain

$$\Omega_\rho = \rho e_N + D_\rho = \{x : \rho/2 < |x - \rho e_N| < \rho\},$$

lies in Ω for all $0 < \rho < \rho_0$.

Following [3] we introduce the auxiliary function $v \in C^{1,\alpha}(\overline{\Omega}_\rho)$ defined as the weak solution to the problem

$$\begin{gathered} \mathcal{L}v + \sum_{i=1}^{N} b_i \partial_i v + cv = 0 \quad x \in \Omega_\rho \\ v = 1 \quad x \in \partial\Omega_\rho^- \\ v = 0 \quad x \in \partial\Omega_\rho^+, \end{gathered} \tag{2.1}$$

where $\partial\Omega_\rho^- = \{x : |x - \rho e_N| = \rho/2\}$ and $\partial\Omega_\rho^+ = \{x : |x - \rho e_N| = \rho\}$. Existence and uniqueness of a positive solution to (2.1) is provided in Lemma 3.1 below.

It is clear that a small $\varepsilon > 0$ can be found so that

$$u - u_0 - \varepsilon v \ge 0,$$

on $\partial\Omega_\rho$ meanwhile

$$\mathcal{L}(u - u_0 - \varepsilon v) + c(u - u_0 - \varepsilon v) \ge 0,$$

in the weak sense in Ω_ρ. The weak maximum principle [5] then implies that $u \ge u_0 + \varepsilon v$, in Ω_ρ. In particular,

$$\frac{\partial u}{\partial \nu}(0) \le \varepsilon \frac{\partial v}{\partial \nu}(0), \tag{2.2}$$

for any outward direction ν to Ω_ρ at $x = 0$. It follows from Lemma 3.4 below that

$$\frac{\partial v}{\partial \nu}(0) \to -\infty \tag{2.3}$$

as $\rho \to 0+$. An even more precise account on the asymptotic behavior of such derivative as $\rho \to 0+$ is given in Lemma 3.4. It is clear that (2.2) and (2.3) imply the desired conclusion (1.5).

Remark 2.1. The following strong comparison principle is stated in [6, Proposition 2.2]. Functions $u, v \in C^1(\overline{\Omega})$, $u = v = 0$ on $\partial\Omega$, solve $-\Delta_p u = f$ and $-\Delta_p v = g$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. It is assumed that $f, g \in L^\infty(\Omega)$, $0 \le f \le g$ while the set $\{x \in \Omega : f(x) = g(x) \text{ a. e.}\}$ has an empty interior. Then $v(x) > u(x)$ for all $x \in \Omega$ together with

$$\frac{\partial v}{\partial n} < \frac{\partial u}{\partial n}, \tag{2.4}$$

at every point in $\partial\Omega$.

As for its proof, by the contradiction argument employed in [6] it follows that $v > u$ in Ω. This is achieved by using weak comparison [12] and the strong maximum principle [13], the latter implying that $\frac{\partial v}{\partial n} < 0$ on $\partial\Omega$. Authors in [6] then obtain (2.4) from the strict inequality between u and v in Ω.

However, we think that to attain (2.4) a more work is required and propose the following argument. Fix $x_0 \in \partial\Omega$ and assume that contrary to (2.4) the equality

$$\frac{\partial v}{\partial n}(x_0) = \frac{\partial u}{\partial n}(x_0) \tag{2.5}$$

holds. Then there exists a small ball B, centered at x_0, such that

$$\min\{|\nabla u(x)|, |\nabla v(x)|\} \ge k > 0$$

in $U := B \cap \Omega$. Thus, the difference $w = v - u$ solves in U an elliptic equation of the form (1.1) with the uniform elliptic matrix

$$A(x) = \int_0^1 |\nabla w_t|^{p-2}\Big(I + (p-2)\frac{\nabla w_t}{|\nabla w_t|} \otimes \frac{\nabla w_t}{|\nabla w_t|}\Big)\, dt, \tag{2.6}$$

where $w_t = (1-t)u + tv$, I is the identity matrix and for $\xi \in \mathbb{R}^N$, $(\xi \otimes \xi)_{ij} = \xi_i \xi_j$, $1 \le i, j \le N$. Since $\frac{\partial v}{\partial n}(x_0) \ne 0$ this implies, by reducing B if necessary, that $0 \notin [\nabla u(x), \nabla v(x)]$ for all $x \in \overline{U}$. Equivalently, that $|\nabla w_t(x)| > 0$ for all $x \in \overline{U}$, $t \in [0,1]$. Taking into account that $u, v \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ [8] then the coefficients a_{ij} of matrix A in (2.6) belong to $C^\alpha(\overline{U})$. In this respect it should be remarked that $\nabla u(x) \ne 0$ and $\nabla v(x) \ne 0$ in $\overline{U}$ are not enough to ensure us that $a_{ij} \in C^\alpha(\overline{U})$. Finally, Theorem 1.1 can now be used to conclude that (2.5) is not possible. Hence, (2.4) holds at x_0.

3. Auxiliary results

Lemma 3.1. *Problem* (2.1) *admits a unique positive solution* $v \in C^{1,\alpha}(\overline{\Omega}_\rho)$.

Proof. Existence of a unique weak solution $v \in H^1(\Omega_\rho)$ to (2.1) is standard [5, Theorem 8.3], being the uniqueness consequence of the weak maximum principle. Just this result implies that $0 \le v \le 1$ a. e. in Ω. Since $v \in L^\infty(\Omega)$, classical results in [7] imply that $v \in C^\beta(\overline{\Omega}_\rho)$ for some $0 < \beta < 1$. Furthermore, strong maximum principle [5, Theorem 8.19] ensures us that $v(x) > 0$ for all $x \in \Omega_\rho$. Also the results in [5, Section 8.11] permit us concluding that $v \in C^{1,\alpha}(\overline{\Omega}_\rho)$. $\square$

Remark 3.2. When $b_i \equiv 0$, $1 \le i \le N$, in (2.1) existence of a weak solution can be directly obtained by a variational argument. In fact, the functional

$$J(u) = \frac{1}{2}\int_{\Omega_\rho} \Big\{ \sum_{i,j=1}^{N} a_{ij}\partial_i u \partial_j u + cu^2 \Big\},$$

is coercive in $\mathcal{M} = \{u \in H^1(\Omega_\rho) : u = \varphi \text{ on } \partial\Omega_\rho\}$, where φ is the characteristic function of $\partial\Omega_\rho^-$ in $\partial\Omega_\rho$. It therefore admits a global minimizer $u \in H^1(\Omega_\rho)$ in $\mathcal{M}$. Moreover, such minimizer is unique due to the convexity of J ($c \ge 0$).

Consider now the elliptic operator

$$\overline{\mathcal{L}}u = -\sum_{i,j=1}^{N} \bar{a}_{ij}\partial_{ij}u,$$

where the coefficients $\bar{a}_{ij}$ are constant and the matrix $\bar{A} = (\bar{a}_{ij})$ is symmetric and positive definite with eigenvalues

$$0 < \bar{\lambda}_1 \le \dots \le \bar{\lambda}_N.$$

Let D be the unitary annulus introduced above and D_ρ the corresponding annulus with exterior radius ρ. Set $G_\rho(x,y)$ the Green function associated to $\overline{\mathcal{L}}$, under homogeneous Dirichlet conditions in D_ρ (see [2]). Namely, the unique function $G_\rho \in C^2(\overline{D}_\rho \times \overline{D}_\rho \setminus \Delta)$, $\Delta = \{(x,x) : x \in \overline{D}_\rho\}$, such that the classical solution $u \in C^2(D_\rho) \cap C(\overline{D}_\rho)$ to the problem

$$\begin{gathered} \overline{\mathcal{L}}u = f \quad x \in D_\rho \\ u = 0 \quad x \in \partial D_\rho, \end{gathered} \tag{3.1}$$

with $f \in C(D_\rho) \cap L^1(D_\rho)$, provided that it exists, can be represented in the form

$$u(x) = \int_{D_\rho} G_\rho(x,y) f(y)\, dy. \tag{3.2}$$

The next result provides us with the key estimates on the derivatives of G_ρ.

Lemma 3.3. *There exist positive constants C_1, C_2 such that*

$$|\partial_{x_i} G_\rho(x,y)| \le \frac{C_1}{|x-y|^{N-1}} \quad 1 \le i \le N, \tag{3.3}$$

$$|\partial_{x_i}\partial_{y_j} G_\rho(x,y)| \le \frac{C_2}{|x-y|^{N}} \quad 1 \le i,j \le N, \tag{3.4}$$

for all $x, y \in D_\rho$, $x \ne y$. Moreover, constants C_1 and C_2 can be estimated as follows:

$$C_1 \le K_1\Big(\frac{\bar{\lambda}_N}{\bar{\lambda}_1}\Big)^{\frac{N-1}{2}}\frac{1}{\bar{\lambda}_1}, \qquad C_2 \le K_2\Big(\frac{\bar{\lambda}_N}{\bar{\lambda}_1}\Big)^{\frac{N}{2}}\frac{1}{\bar{\lambda}_1}, \tag{3.5}$$

where the positive constants K_1 and K_2 do not depend on ρ.

Proof. There exists a linear isomorphism $y = Tx$ which maps D_ρ into the ellipsoidal cavity $\mathcal{D}_\rho = \{\rho y : y \in \mathcal{D}\}$ with

$$\mathcal{D} = \{y \in \mathbb{R}^N : \frac{1}{4} < \sum_{i=1}^{N} \frac{y_i^2}{a_i^2} < 1\},$$

and where the reference semiaxis a_i are given by

$$a_i = \frac{1}{\sqrt{\lambda_i}} \quad i = 1, \dots, N.$$

Moreover, T transforms problem (3.1) into

$$\begin{gathered} -\Delta v = g \quad y \in \mathcal{D}_\rho \\ v = 0 \quad y \in \partial\mathcal{D}_\rho, \end{gathered} \tag{3.6}$$

where $v(y) = u(T^{-1}y)$, $g(y) = f(T^{-1}y)$. Let $\widetilde{G}_\rho = \widetilde{G}_\rho(y,\eta)$ be the Green function associated to $-\Delta$ under homogeneous Dirichlet conditions in $\mathcal{D}_\rho$. A direct computation shows that

$$G_\rho(x,\xi) = \{\det T\}\ \widetilde{G}_\rho(Tx, T\xi),$$

for all $x,\xi \in D_\rho$, $x \neq \xi$, where $\det T = a_1 \cdots a_N$. A further scaling argument permits us writing

$$\widetilde{G}_\rho(y,\eta) = \rho^{2-N} G\Big(\frac{y}{\rho}, \frac{\eta}{\rho}\Big), \quad y,\eta \in \mathcal{D}_\rho,\ y \neq \eta,$$

$G = G(z,\zeta)$ being the Green function for $-\Delta$, constrained with homogeneous Dirichlet conditions in $\mathcal{D}$. Therefore,

$$G_\rho(x,\xi) = \rho^{2-N}\{\det T\} G\Big(\frac{Tx}{\rho}, \frac{T\xi}{\rho}\Big) \quad x,\xi \in D_\rho \quad x \neq \xi.$$

Now, the estimates in [14] allow us assert the existence of a positive constant M such that

$$\begin{gathered} |\partial_{x_i} G(x,y)| \le \frac{M}{|x-y|^{N-1}} \quad 1 \le i \le N, \\ |\partial_{x_i}\partial_{y_j} G(x,y)| \le \frac{M}{|x-y|^{N}} \quad 1 \le i,j \le N, \end{gathered} \tag{3.7}$$

for all $x,y \in \mathcal{D}$, $x \neq y$. Next we observe that isomorphism T can be chosen of the form

$$T = \operatorname{diag}(a_1, \dots, a_N)\ L,$$

where L is an orthogonal transformation. Thus,

$$\partial_{x_i} G_\rho(x,y) = \rho^{1-N} \sum_{k=1}^{N} \partial_{z_k} G\Big(\frac{Tx}{\rho}, \frac{Ty}{\rho}\Big) \partial_{x_i}((Tx)_k),$$

where $(Tx)_k = a_k \sum L_{xs} x_s$. Then, the estimate

$$\sum_{k=1}^{N} |\partial_{x_i}((Tx)_k)| \le \sqrt{N} a_1$$

follows easily. In addition,

$$|Tx| = |\operatorname{diag}(a_1, \dots, a_N) L x| \ge a_1 |x|,$$

for all $x \in \mathbb{R}^N$. By (3.7) with the last estimates, the first inequality in (3.5) is obtained with the choice

$$K_1 = M\sqrt{N}.$$

By proceeding in the same way, the second inequality in (3.5) holds for $K_2 = MN$. □

Lemma 3.4. *Let $v \in C^{1,\alpha}(\overline{\Omega}_\rho)$ be the positive solution of the problem (2.1). Then,*

$$\frac{\partial v}{\partial \nu}(0) \sim \frac{C_N^*}{\rho}\langle \nu, e_N\rangle \quad as \quad \rho \to 0+, \tag{3.8}$$

where $\nu \in \mathbb{R}^N$ is any unitary vector and

$$C_N^* = \frac{N-2}{2^{N-2}-1}.$$

Remark 3.5. Observe that exterior directions ν to Ω_ρ at $x = 0$ are characterized by $\langle \nu, e_N\rangle < 0$.

Proof of Lemma 3.4. To prove (3.8) we follow the argument in [3] and introduce $u = \psi$, the solution of the problem

$$\begin{gathered} \Delta u = 0 \quad x \in \Omega_\rho \\ u = 1 \quad x \in \partial\Omega_\rho^- \\ u = 0 \quad x \in \partial\Omega_\rho^+, \end{gathered}$$

whose explicit form is

$$\psi(x) = \Big(\frac{1}{|x-\rho e_N|^{N-2}} - \frac{1}{\rho^{N-2}}\Big)\frac{\rho^{N-2}}{2^{N-1}-1}.$$

We fix now $\bar{x} \in \Omega_\rho$ and define the constant coefficients operator

$$\mathcal{L}_{\bar{x}} u := -\sum_{i,j=1}^N \bar{a}_{ij}\partial_{ij} u,$$

with $\bar{a}_{ij} = a_{ij}(\bar{x})$. By noticing that $w(x) := v(x) - \psi(x)$ vanishes at the boundary $\partial\Omega_\rho$ of Ω_ρ, w can be represented as

$$w(x) = \int_{\Omega_\rho} G_\rho(x,y)\mathcal{L}_{\bar{x}} w(y)\,dy, \tag{3.9}$$

where G_ρ stands for the Green function of the operator $\mathcal{L}_{\bar{x}}$ in Ω_ρ, subject to homogeneous Dirichlet conditions on $\partial\Omega_\rho$ (see Lemma 3.3). We are employing (3.9) to analyze ∇w near zero when ρ becomes small.

Observe that,

$$\begin{aligned} w(x) &= \int_{\Omega_\rho} G_\rho(x,y)(\mathcal{L}_{\bar{x}} v(y) - \mathcal{L} v(y))\,dy \\ &\quad - \int_{\Omega_\rho} G_\rho(x,y)(\mathcal{L}_{\bar{x}}\psi(y) - \mathcal{L}_0\psi(y))\,dy \\ &\quad - \int_{\Omega_\rho} G_\rho(x,y)(b(y)\nabla v(y) + c(y)v(y))\,dy \\ &=: w_1(x) + w_2(x) + w_3(x), \end{aligned}$$

$x \in \Omega_\rho$, with $b = (b_i)$ and where $\mathcal{L}_0$ is the constant coefficients operator resulting from fixing $x = 0$ in the functions $a_{ij}(x)$. Notice that $\mathcal{L}_0 = -\Delta$ and so $\mathcal{L}_0\psi = 0$.

On the other hand,

$$w_1(x) = \sum_{i,j=1}^N \int_{\Omega_\rho} \partial_{y_i} G_\rho(x,y)(a_{ij}(y) - a_{ij}(\bar{x}))\partial_j v(y)\,dy.$$

Hence,

$$\partial_{x_s} w_1(\bar{x}) = \sum_{i,j=1}^{N} \int_{\Omega_\rho} \partial_{x_s}\partial_{y_i} G_\rho(\bar{x}, y)(a_{ij}(y) - a_{ij}(\bar{x}))\partial_j v(y)\, dy. \tag{3.10}$$

By estimate (3.4) in Lemma 3.3,

$$|\partial_{x_s} w_1(\bar{x})| \le \sum_{i,j=1}^{N} C_2[a_{ij}]_\alpha \|\nabla v\|_{\infty,\Omega_\rho} \int_{\Omega_\rho} \frac{1}{|y-\bar{x}|^{N-\alpha}}\, dy,$$

where

$$[a_{ij}]_\alpha = \sup_{x,y\in\Omega, x\neq y} \frac{|a_{ij}(x) - a_{ij}(y)|}{|x-y|^\alpha}.$$

After estimating the integral, (3.10) implies that

$$|\nabla w_1(\bar{x})| \le C\|\nabla v\|_{\infty,\Omega_\rho}\rho^\alpha \quad \bar{x} \in \Omega_\rho, \tag{3.11}$$

for a certain positive constant C which does not depend on ρ. Label C will be employed in the sequel to designate positive constants which no depend on ρ, and whose precise value is irrelevant for the discourse.

As for the gradient of w_2,

$$\partial_{x_s} w_2(\bar{x}) = \sum_{i,j=1}^{N} \int_{\Omega_\rho} \partial_{x_s} G_\rho(\bar{x}, y)(a_{ij}(0) - a_{ij}(\bar{x}))\partial_{ij}\psi(y)\, dy.$$

Since $|\partial_{ij}\psi(y)| \le C\rho^{-2}$, using estimate (3.3) we find that

$$|\partial_{x_s} w_2(\bar{x})| \le \sum_{i,j=1}^{N} C[a_{ij}]_\alpha \rho^{\alpha-2} \int_{\Omega_\rho} \frac{1}{|y-\bar{x}|^{N-1}}\, dy.$$

By estimating the integral in terms of ρ we obtain

$$|\partial_{x_s} w_2(\bar{x})| \le C\rho^{\alpha-1} \quad \bar{x} \in \Omega_\rho. \tag{3.12}$$

On the other hand, taking into account that $v(0) = 0$, we conclude that

$$|\partial_{x_s} w_3(\bar{x})| \le C_1\|c\|_{\infty,\Omega}\|\nabla v\|_{\infty,\Omega_\rho} \int_{\Omega_\rho} \frac{1}{|y-\bar{x}|^{N-1}}\, dy,$$

and so,

$$|\partial_{x_s} w_3(\bar{x})| \le C\|\nabla v\|_{\infty,\Omega_\rho}\rho^2 \quad \bar{x} \in \Omega_\rho. \tag{3.13}$$

From (3.11), (3.12) and (3.13) the estimate

$$\|\nabla w\|_{\infty,\Omega_\rho} \le C\|\nabla v\|_{\infty,\Omega_\rho}\rho^\alpha + C\rho^{\alpha-1} \tag{3.14}$$

holds.

Now, $\|\nabla v\|_{\infty,\Omega_\rho}$ can be estimated in terms of ρ. In fact,

$$|\nabla v(x)| \le |\nabla w(x)| + |\nabla\psi(x)| \le \|\nabla w\|_{\infty,\Omega_\rho} + C\rho^{-1} \quad x \in \Omega_\rho.$$

Hence,

$$\|\nabla v\|_{\infty,\Omega_\rho} \le C\rho^{-1},$$

which, together with (3.14), imply that

$$\|\nabla w\|_{\infty,\Omega_\rho} \le C\rho^{\alpha-1}. \tag{3.15}$$

Finally,

$$\Big|\frac{\partial v}{\partial \nu}(0)-\frac{\partial \psi}{\partial \nu}(0)\Big|=\Big|\frac{\partial v}{\partial \nu}(0)-\frac{C_N^*}{\rho}\langle \nu,e_N\rangle\Big|\le \|\nabla w\|_{\infty,\Omega_\rho}\le C\rho^{\alpha-1}.$$

Thus,

$$\frac{C_N^*}{\rho}\Big(\langle \nu,e_N\rangle-\frac{C}{C_N^*}\rho^\alpha\Big)\le \partial_\nu v(0)\le \frac{C_N^*}{\rho}\Big(\langle \nu,e_N\rangle+\frac{C}{C_N^*}\rho^\alpha\Big),$$

for ρ small. Asymptotic estimate (3.8) immediately follows from these inequalities. □

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