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The tangential variation of a localized flux-type eigenvalue problem

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To the memory of Fuensanta Andreu Vaillo, a gifted mathematician and wonderful person

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A B S T R A C T

In this work the differentiability of the principal eigenvalue $\lambda = \lambda_1(\Gamma)$ to the localized Steklov problem $-\Delta u + qu = 0$ in Ω, $\frac{\partial u}{\partial \nu} = \lambda \chi_\Gamma(x) u$ on $\partial\Omega$, where $\Gamma \subset \partial\Omega$ is a smooth subdomain of $\partial\Omega$ and χ_Γ is its characteristic function relative to $\partial\Omega$, is shown. As a key point, the flux subdomain Γ is regarded here as the variable with respect to which such differentiation is performed. An explicit formula for the derivative of $\lambda_1(\Gamma)$ with respect to Γ is obtained. The lack of regularity up to the boundary of the first derivative of the principal eigenfunctions is a further intrinsic feature of the problem. Therefore, the whole analysis must be done in the weak sense of $H^1(\Omega)$. The study is of interest in mathematical models in morphogenesis.

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1. Introduction

In this work we are analyzing the flux-type linear eigenvalue problem,

$$\begin{cases} -\Delta u + q(x)u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda \chi_\Gamma(x) u, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a class C^3 bounded domain with boundary $\partial\Omega$ and outer unit normal ν. As an important feature to be pointed out, the weight function $\chi_\Gamma(x)$ in front of λ is the characteristic function of a region Γ in $\partial\Omega$ ($\chi_\Gamma = 1$ if $x \in \Gamma$, $\chi_\Gamma = 0$ for $x \in \partial\Omega \setminus \Gamma$). Throughout this work, it will be always assumed that Γ is a subdomain (an open connected set) so that $\overline{\Gamma} = \Gamma \cup \partial\Gamma$ defines a class C^3 closed submanifold of $\partial\Omega$ with boundary $\partial\Gamma$. Thus, $\partial\Gamma$ is also a closed $N-2$ dimensional manifold. We will refer to this requirement of the flux region Γ in the sequel by saying that Γ is a *smooth subdomain* of $\partial\Omega$. In addition, the potential term q will be supposed C^1 up to the boundary, i.e. $q \in C^1(\overline{\Omega})$.

The main objective of this paper is to show that the principal eigenvalue to problem (1.1) varies in a smooth way when the flow region Γ is "tangentially" deformed according to a broad class of regular perturbations (see (4.1) and Section 3 for precise definitions). Furthermore, an explicit formula for the variation of such eigenvalue with respect to Γ is obtained (Section 4, Theorem 4.2 and Section 5, Theorem 5.1). Accordingly, the perturbation problem addressed here falls in the realm of "variation of domains", a field with long tradition in the theory of linear and nonlinear eigenvalue problems (see the specific monographies [22,14,13,15] on the subject, and [24,19] together with its references).

Problem (1.1) can be observed as a Steklov eigenvalue problem where the flux through the boundary is restricted, by means of the weight function χ_Γ, to a specific zone Γ of $\partial\Omega$ (see [25,13] for the classical Steklov problem, further references in [23], and recent results on related problems in [2,17,6, 4,11]). Our main interest will be focused on *principal eigenvalues* to (1.1), i.e. eigenvalues λ with a positive associated eigenfunction Φ. Precise conditions characterizing the existence of a *unique* principal eigenvalue λ_1 are stated in Section 2 (see Theorem 2.1).

The principal eigenvalue plays a crucial role when one deals with natural perturbations of (1.1) and the interest is put in positive solutions. Specifically, consider the problem,

$$\begin{cases} -\Delta u + q(x)u = f(x,u), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \chi_\Gamma(x)\big(\lambda u + g(x,u)\big), & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ define certain volumetric and surface reaction terms, respectively. Assume that $f(x,u) = u f_1(x,u)$, $g(x,u) = u g_1(x,u)$ with both f_1 and g_1 continuously differentiable and satisfying $f_1(x,0) = g_1(x,0) = 0$ in Ω. Then problem (1.2) can be regarded as a model for a chemical reactor Ω where the species u is consumed in a rate $-q + f_1$ meanwhile it is pumped into the reactor with a flux-intensity λ through the window Γ in the boundary $\partial\Omega$ (see [12] for related ideas). In fact, a positive solution u to (1.2) – if such a solution exists – provides the equilibrium regime of production for such a substance u.

Suppose now that both f_1 and g_1 are decreasing. A simple computation reveals that a necessary condition for the existence of such a positive solution is that the intensity λ be greater than λ_1. Furthermore, $\lambda > \lambda_1$ turns out to be also a sufficient condition for the existence of a unique positive equilibrium provided $f_1(x,u) \to -\infty$, $g_1(x,u) \to -\infty$ as $u \to \infty$ (see [11] for precise details together with further configurations for the reaction terms f and g). This means that the system requires a large enough flux intensity λ through the "localized zone" Γ, to sustain a stable regime. The critical value of λ is just provided by λ_1. On the other hand, $\lambda = \lambda_1$ constitutes a bifurcation value, either from zero or infinity, for positive solutions of (1.2) if suitable structure conditions are satisfied by the nonlinearities f and g (see [4,5]).

In [10] authors presented a reaction–diffusion model for patterning of limb cartilage development, a paramount problem in embryology ([20]). They considered a growing domain modeling the limb bud (the reactor Ω), and developed a numerical scheme that incorporated the interactions between two distinguished reactants u_1, u_2 located in very specific zones Γ_1, Γ_2 of the boundary $\partial\Omega$. The relevance of such substances u_i (called *morphogens*) and the prominent role of the flux regions Γ_i has been largely supported by a strong experimental evidence ([21,27]). Experiments also suggests that the pattern-formation seems to be driven by the mutual regulation of the fluxes of u_i through the zones Γ_i.

Inspired in [10] the present work analyzes the phenomenology of the flux zones from an alternative point of view. Since λ_1 measures the threshold value of λ in order that (1.2) exhibits a positive solution, a special emphasis should be put on how does λ_1 varies with Γ. Therefore, the "size" of the region $\Gamma \subset \partial\Omega$ will be regarded here as a parameter in the sense that the whole of Γ will be subject to tangential deformations. Our main purpose will be then to study the corresponding variations of λ_1, as direct response to such perturbation.

It should be stressed that in order that (1.1) generates a "genuine" perturbation problem when the subdomain Γ is varied, it is required that $\lambda_1^{\mathcal{N}}(q) \neq 0$, $\lambda_1^{\mathcal{N}}(q)$ being the first Neumann eigenvalue of $-\Delta + q$ in Ω. Otherwise, the principal eigenvalue λ_1 to (1.1) stays equal to zero for all subdomains $\Gamma \subset \partial\Omega$ and the perturbation problem becomes degenerate (Section 2, Remarks 2.3 and 2.5).

Another key feature of problem (1.1) is the lack of regularity exhibited by the eigenfunctions associated to the principal eigenvalue λ_1. In fact, such eigenfunctions fails to be of class C^1 up to the boundary (Section 2, Theorem 2.1). This singular behavior is caused by the discontinuity of the coefficient χ_Γ through the interphase $\partial\Gamma$ (the boundary of $\partial\Gamma$ in $\partial\Omega$). As a direct consequence of this fact, the full analysis of existence of λ_1 and its continuity and differentiability with respect to Γ must be necessarily performed in the "weak" framework of $H^1(\Omega)$.

The present work is organized as follows. Section 2 is devoted to a complete study of the principal eigenvalue to (1.1) which covers existence conditions, uniqueness, simplicity and regularity of eigenfunctions (Theorem 2.1). Monotone and continuous dependence with respect to weak perturbations of the subdomain Γ are also studied (Lemmas 2.4 and 2.6). In addition, a Fredholm alternative result for λ_1, which is necessary for the analysis in Section 4, is directly shown by following a variational approach (Theorem 2.7). Section 3 lays down the class of smooth perturbations of Γ under which the smoothness of λ_1 is studied. It also contains the relevant calculus features required for our purposes. Finally, the main results of the work are contained in Sections 4 and 5. Namely, the differentiability of λ_1 with respect to Γ (Theorem 4.1) and an explicit integral formula for its derivative (Theorems 4.2 and 5.1).

2. The localized Steklov eigenvalue problem

For the immediate purposes of this work we are considering a slightly more general problem than (1.1). Namely,

$$\begin{cases} -\Delta u + q(x)u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda m(x) u, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $m \in L^\infty(\Gamma)$, $m = 0$ in $\partial\Omega \setminus \Gamma$ and $m > 0$ a.e. in Γ. An eigenvalue $\lambda \in \mathbb{R}$ to (2.1) with an associated (weak) eigenfunction $\Phi \in H^1(\Omega) \setminus \{0\}$ is defined through the equality

$$\int_\Omega \nabla\Phi\nabla\varphi + q\Phi\varphi = \lambda \int_\Gamma m\Phi\varphi, \tag{2.2}$$

which must be satisfied for every $\varphi \in H^1(\Omega)$. A first result is the following.

Theorem 2.1. *A necessary and sufficient condition for the existence of a principal eigenvalue to* (2.1) *is*

$$\mu_1 > 0, \tag{2.3}$$

where $\mu = \mu_1$ is the principal eigenvalue of the mixed *problem,*

$$\begin{cases} -\Delta\phi + q\phi = \mu\phi & \text{in } \Omega, \\ \phi = 0 \quad \text{on } \Gamma \quad \& \quad \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega \setminus \overline{\Gamma}. \end{cases} \tag{2.4}$$

Moreover, if (2.3) *holds*:

- i) *Problem* (2.1) *possesses a minimum eigenvalue λ_1 which is the* unique *principal eigenvalue to* (2.1)*. Moreover, λ_1 is simple.*
- ii) *The sign of λ_1 coincides with the sign of the first Neumann eigenvalue of $-\Delta + q$ in Ω. In particular, $\lambda_1 = 0$ if such eigenvalue is zero.*
- iii) *If $\Phi_1 \in H^1(\Omega)$ is any eigenfunction associated to λ_1 then $\Phi_1 \in C^{2,\alpha}(\Omega) \cap L^\infty(\Omega)$. Moreover, $\Phi_1 \in C^\beta(\overline{\Omega})$ for certain $0 < \beta < 1$. In addition, if $m \in C^{1,\alpha}(\partial\Omega)$ then $\Phi_1 \in C^{2,\alpha}(\overline{\Omega})$.*
- iv) *For the special choice $m = \chi_\Gamma$ and Φ_1 as in* iii)*, $\Phi_1 \in C^{2,\alpha}(\Omega \cup K)$ for every compact $K \subset \partial\Omega$, $K \cap \partial\Gamma = \emptyset$. Furthermore, Φ_1* cannot be *continuously differentiable up to the boundary $\partial\Omega$.*
- v) *If $\Phi_1 \in H^1(\Omega)$ is a nonnegative principal eigenfunction to* (2.1) *then $\Phi_1 > 0$ in $\overline{\Omega}$, in particular on the boundary $\partial\Gamma$ of $\overline{\Gamma}$ as a manifold with boundary.*

Definition 2.2. Provided (2.3) holds, λ_1 ($\lambda_1(\Gamma)$ if it is necessary to emphasize the dependence of λ_1 on Γ) will designates the principal eigenvalue to (1.1). Likewise, $\Phi_1 \in H^1(\Omega)$ will stands for the associated positive eigenfunction such that $\int_\Gamma \Phi_1^2 = 1$.

Remark 2.3.

a) Theorem 2.1 remains valid if Γ is merely a relative open subdomain of $\partial\Omega$ rather than a smooth subdomain of $\partial\Omega$. On the other hand, it follows from the variational characterization of μ_1 that

$$\lambda_1^{\mathcal{N}}(q) < \mu_1 < \lambda_1^{\mathcal{D}}(q), \tag{2.5}$$

for all $\Gamma \subset \partial\Omega$, $\Gamma \neq \partial\Omega$, where $\lambda_1^{\mathcal{N}}(q)$ and $\lambda_1^{\mathcal{D}}(q)$ stand for the principal eigenvalues of $-\Delta + q$ under Neumann or Dirichlet boundary conditions in Ω, respectively. It is a consequence of Theorem 2.1 and (2.5) that $\lambda_1^{\mathcal{D}}(q) > 0$ becomes a necessary condition for the existence of a principal eigenvalue λ_1 to (1.1). It also provides the existence of λ_1 at least for certain subdomains Γ (see also Remark 2.5).

b) A crucial consequence of ii) is the fact that condition $\lambda_1^{\mathcal{N}}(q) \neq 0$ is required in order that our problem of perturbing λ_1 with respect to Γ be a nontrivial problem (otherwise $\lambda_1(\Gamma)$ vanishes for all subdomains Γ of $\partial\Omega$).

Proof of Theorem 2.1. The fact that (2.4) possesses a unique principal eigenvalue $\mu = \mu_1$ is essentially well known and can be proved by direct methods in the calculus of variations. Moreover, μ_1 is unique as a principal eigenvalue and can be variationally expressed as

$$\mu_1 := \inf_{H^1_\Gamma(\Omega)} \frac{\int_\Omega |\nabla u|^2 + qu^2}{\int_\Omega u^2}, \tag{2.6}$$

where $H^1_\Gamma(\Omega) := \{u \in H^1(\Omega)\colon\ u_{|\Gamma} = 0\}$.

Let us first assume that $\mu_1 > 0$. We are going to show the existence and remaining properties of a principal eigenvalue to (2.1) by proving that

$$\lambda_1 := \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + qu^2}{\int_\Gamma mu^2} > -\infty \tag{2.7}$$

and that such infimum is achieved at some $\Phi_1 \in H^1(\Omega) \setminus \{0\}$. In fact, the functional $J(u) := \int_\Omega |\nabla u|^2 + qu^2$ is sequentially weakly lower semicontinuous in $H^1(\Omega)$. We claim that J is also coercive on $\mathcal{M} := \{u \in H^1(\Omega)\colon \int_\Gamma mu^2 = 1\}$ provided $\mu_1 > 0$. Therefore, a standard approach in the calculus of variations shows the existence of $\Phi_1 \in \mathcal{M}$ such that $J(\Phi_1) = \inf_{u \in \mathcal{M}} J(u) := \lambda_1$.

To prove the claim, let us consider $u_n \in \mathcal{M}$, $\|u_n\|_{H^1(\Omega)} \to \infty$. Then $J(u_n) \to \infty$. Otherwise, by setting $u_n = t_n v_n$ with $t_n = \|u_n\|_{H^1(\Omega)}$ we obtain that $J(v_n) = O(t_n^{-2})$. Since, modulus a subsequence, $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$ then $v_n \to v$ both in $L^2(\Omega)$ and $L^2(\partial\Omega)$. However, $\int_\Gamma m v_n^2 = t_n^{-2}$ and so $v \in H^1_\Gamma(\Omega)$. By taking 'lim–inf' in the previous expression for $J(v_n)$ it follows that $J(v) \leqslant 0$. Condition $\mu_1 > 0$ implies that $v = 0$ and we deduce $\int_\Omega |\nabla v_n|^2 = o(1)$. Thus $v_n \to 0$ in $H^1(\Omega)$ which contradicts $\|v_n\|_{H^1(\Omega)} = 1$ for all n. Hence, J is coercive.

On the other hand, it is clear that any $\Phi \in \mathcal{M}$ such that $J(\Phi) = \lambda_1$ defines a weak eigenfunction associated to λ_1 in the sense of (2.2), and so λ_1 is an eigenvalue. Additionally, from (2.2), any other possible eigenfunction $\tilde{\Phi}$ associated to λ_1 satisfies $\int_\Omega |\nabla \tilde{\Phi}|^2 + q\tilde{\Phi}^2 = \lambda_1 \int_\Gamma m\tilde{\Phi}^2$. Since $\mu_1 > 0$ then $\tilde{\Phi} \neq 0$ on Γ. Thus, we get

$$\lambda_1 = \frac{\int_\Omega |\nabla \tilde{\Phi}|^2 + q\tilde{\Phi}^2}{\int_\Gamma m\tilde{\Phi}^2}. \tag{2.8}$$

To show that λ_1 defines a principal eigenvalue notice that if Φ is an eigenfunction associated to λ_1 then $\tilde{\Phi} = |\Phi|$ also satisfies (2.8). Hence $|\Phi| \in H^1(\Omega)^+$ defines an eigenfunction. In addition, the regularity theory for elliptic equations implies that $|\Phi| \in C^{2,\alpha}(\Omega)$ which, together with the maximum principle yields $|\Phi(x)| > 0$ in Ω.

We next show the simplicity of λ_1. It suffices with proving that any eigenfunction Φ associated to λ_1 is one signed ([17]). Assume that, say, $\Phi^+ \neq 0$ on Γ then, by inserting $\varphi = \Phi^+$ as a test function in Eq. (2.2) for Φ we obtain that Φ^+ also satisfies (2.8) with $\tilde{\Phi} = \Phi^+$. This means that Φ^+ is an eigenfunction and, as already shown, $\Phi^+(x) > 0$ in Ω what says that $\Phi^- = 0$. Therefore, Φ is one signed.

The uniqueness of λ_1 as a principal eigenvalue is a consequence of the fact that $\Phi \neq 0$ on Γ for any other eigenfunction Φ associated to any eigenvalue λ to (2.1). If $\lambda \neq \lambda_1$ and Φ_1 is an eigenfunction associated to λ_1 one easily finds

$$\int_\Gamma \Phi\Phi_1 = 0.$$

This is impossible if $\Phi \neq 0$ is nonnegative. Observe in addition that the own expression (2.7) entails the minimality of λ_1 as an eigenvalue of (2.1).

Let us consider now the regularity issues. If $\Phi \in H^1(\Omega)$ is any (not necessarily principal) eigenfunction to (2.1) then Lemma 5 in [11] (see also [3]) allows us ensuring that $\Phi \in L^\infty(\Omega)$. Moreover, the existence of $\beta \in (0,1)$ such that $\Phi \in C^\beta(\overline{\Omega})$ follows from Lemma B.1 in [3]. That $\Phi \in C^{2,\beta}(\Omega \cup K)$ for any compact $K \subset \partial\Omega \setminus \partial\Gamma$ and every m supported in Γ provided $m \in C^{1,\alpha}(\Gamma)$ follows from classical regularity theory ([1]). Of course, $\Phi \in C^{2,\alpha}(\overline{\Omega})$ if $m \in C^{1,\beta}(\partial\Omega)$ for $\beta \geqslant \alpha$.

However, when $m(x) = \chi_\Gamma(x)$ – which is just our main concern in this work – a principal eigenfunction Φ_1 cannot be continuously differentiable up to the boundary. In fact, supposing $\Phi_1 > 0$ in Ω then Φ_1 must be positive on $\partial\Omega \setminus \partial\Gamma$. If $\Phi_1 \in C^1(\overline{\Omega})$ then $\partial\Phi_1/\partial\nu$ should be zero at $\partial\Gamma$ and the same should be true for Φ_1. But this contradicts Hopf's maximum principle and so the normal derivative cannot be continuous through $\partial\Gamma$. Moreover, we are showing below that $\Phi_1 > 0$ on $\partial\Gamma$. Hence $\partial\Phi_1/\partial\nu$ undergoes a jump discontinuity across $\partial\Gamma$.

In conclusion, we have completed the proofs of i), iii) and iv).

To show the necessity of (2.3), let us introduce the auxiliary eigenvalue problem

$$\begin{cases} -\Delta\phi + q\phi = \theta\phi, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = \lambda m(x)\phi, & x \in \partial\Omega, \end{cases} \tag{2.9}$$

with $m \in L^\infty(\partial\Omega)^+$, supported on Γ. In [11] (see early results in [17]) it has been shown the existence, for each $\lambda \in \mathbb{R}$, of a unique principal eigenvalue $\theta = \theta(\lambda)$ to (2.9), with a nonnegative associated eigenfunction $\phi \in H^1(\Omega)$. Furthermore it follows from the proof of Lemma 8 in [11] that function $\theta(\lambda)$ is concave, decreasing and that $\lim_{\lambda\to\infty}\theta(\lambda) = -\infty$. In addition, we claim that

$$\lim_{\lambda\to-\infty}\theta(\lambda) = \mu_1.$$

Then, if λ is a principal eigenvalue to (2.1) this means that $\theta(\lambda)$ is zero. Therefore, $\mu_1 > 0$ since otherwise $\theta(\lambda)$ never vanishes.

Let us sketch the proof of the claim and choose $\phi_n \in H^1(\Omega)$, $\int_\Omega \phi_n^2 = 1$ a positive eigenfunction to (2.9) associated to $\theta_n := \theta(\lambda_n)$, with λ_n decreasing to $-\infty$. Then the variational characterization of θ ([11]) both implies that $\theta(\lambda_n) \leqslant \mu_1$ for all n and that $\|\phi_n\|_{H^1(\Omega)}$ stays bounded. Hence, modulus a subsequence, $\phi_n \rightharpoonup \phi$ weakly in $H^1(\Omega)$ and it is found that $\phi \in H^1_\Gamma(\Omega)$. By putting $\theta^* = \sup\theta_n$ it can be checked that ϕ is a principal eigenfunction to (2.4) associated to $\mu = \theta^*$. Thus $\theta^* = \mu_1$ follows from the uniqueness of μ_1 as a principal eigenvalue to (2.4) and the proof of the claim is finished.

On the other hand, that λ_1 and $\lambda_1^{\mathcal{N}}(q)$ (the principal Neumann eigenvalue of $-\Delta + q$) share sign derives from the fact that $\lambda_1 > 0$ (respectively, $\lambda_1 < 0$) if and only if $\theta(0) = \lambda_1^{\mathcal{N}}(q)$ is positive (negative). In addition, $\lambda_1 = 0$ for all $\Gamma \subset \partial\Omega$ if $\lambda_1^{\mathcal{N}}(q) = 0$.

Let us show now the positivity *up to the boundary* of a principal eigenfunction Φ_1 which is positive in Ω (point v)). To this purpose we first assume that $\lambda_1 > 0$ and observe that $\bar{u} = \Phi_1$ defines a weak supersolution to the problem

$$\begin{cases} -\Delta u + qu = 0 & \text{in } \Omega \setminus B, \\ u = c \quad \text{on } \partial B \quad \& & \dfrac{\partial u}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

where $B \subset \overline{B} \subset \Omega$ is any fixed open ball, $c = \inf_{\partial B}\Phi_1 > 0$. Existence and uniqueness of a weak (and therefore classic) solution $u \in C^{2,\alpha}(\overline{\Omega})$ to (2.10) is consequence of the existence and positiveness of the first eigenvalue $\tilde{\mu} = \tilde{\mu}_1$ to the problem

$$\begin{cases} -\Delta u + qu = \tilde{\mu}u & \text{in } \Omega \setminus B, \\ u = 0 \quad \text{on } \partial B \quad \& & \dfrac{\partial u}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

In fact, existence of $\tilde{\mu}_1$ can be achieved by the variational arguments already discussed in the course on this proof. Moreover,

$$\tilde{\mu}_1 = \inf\frac{\int_{\Omega\setminus B}\{|\nabla u|^2 + qu^2\}}{\int_{\Omega\setminus B} u^2},$$

the infimum being extended to those $u \in H^1(\Omega \setminus B)$ which vanish on ∂B. From this characterization it follows that $\tilde{\mu}_1 \geqslant \lambda_1^{\mathcal{N}}(q)$ while $\lambda_1^{\mathcal{N}}(q) > 0$ due to the assumption $\lambda_1 > 0$. Thus $\tilde{\mu}_1 > 0$ and so $\Phi_1 \geqslant u$ in Ω, with u the solution to (2.10). On the other hand, classical maximum principle implies that $u > 0$ in $\overline{\Omega}$. Therefore, the same happens to Φ_1.

For the case $\lambda_1 < 0$ we observe in turn that $\bar{u} = \Phi_1$ constitutes a supersolution to the alternative problem

$$\begin{cases} -\Delta u + qu = 0 & \text{in } \Omega \setminus B, \\ u = c \quad \text{on } \partial B \quad \& \quad \dfrac{\partial u}{\partial \nu} = \lambda_1 m(x) u & \text{on } \partial\Omega, \end{cases} \tag{2.12}$$

with B and c as above. Existence and uniqueness of a classical solution u to (2.12), $u > 0$ in $\overline{\Omega}$, together with the inequality $\Phi_1 \geqslant u$ in Ω can be shown by the same kind of arguments as in the case $\lambda_1 > 0$ (details are omitted by the sake of brevity). This concludes the proof of v). $\square$

Let us briefly discuss now the monotonicity properties of the principal eigenvalue $\lambda_1(\Gamma)$ to (1.1) as a function of Γ. Accordingly, set $\mu = \mu_1(\Gamma)$ and $\theta = \theta(\lambda, \Gamma)$ the principal eigenvalues to the auxiliary problems (2.4) and (2.9), respectively, where the choice $m = \chi_\Gamma$ has been performed in (2.9). From the variational expression for θ it follows that if $\Gamma \subset \Gamma' \subset \partial\Omega$, $\Gamma \neq \Gamma'$, then

$$\theta(\lambda, \Gamma) < \theta\big(\lambda, \Gamma'\big)$$

provided $\lambda < 0$, meanwhile the reverse strict inequality holds if $\lambda > 0$ (see a detailed analysis in [11]). Similarly, $\mu_1(\Gamma) < \mu_1(\Gamma')$ holds under the same conditions for Γ, Γ'. Our next statement is a direct consequence of these reflections.

Lemma 2.4. *Let $\Gamma \subsetneq \Gamma'$ be smooth nonempty strict subdomains of $\partial\Omega$. Assume*

$$\mu_1(\Gamma) > 0.$$

If the first Neumann eigenvalue $\lambda_1^{\mathcal{N}}(q) < 0$, then $\lambda_1(\Gamma) < \lambda_1(\Gamma')$, while the reverse inequality holds true if $\lambda_1^{\mathcal{N}}(q) > 0$.

Remark 2.5.

a) Since the signs of $\lambda_1(\Gamma)$ and $\lambda_1^{\mathcal{N}}(q)$ (whenever $\lambda_1(\Gamma)$ is defined) coincide, Lemma 2.4 says that $\lambda_1(\Gamma)$ increases with Γ if $\lambda_1(\Gamma) < 0$ while it decreases with Γ if $\lambda_1(\Gamma) > 0$. On the other hand, if $\lambda_1(\Gamma)$ vanishes for some Γ this means that $\lambda_1^{\mathcal{N}}(q) = 0$ and hence $\lambda_1(\Gamma) = 0$ for all $\Gamma \subset \partial\Omega$.
b) It follows from (2.5) that $\lambda_1(\Gamma)$ is defined for all $\Gamma \subset \partial\Omega$ provided $\lambda_1^{\mathcal{N}}(q) > 0$. On the other hand, if $\lambda_1^{\mathcal{N}}(q) < 0 < \lambda_1^{\mathcal{D}}(q)$, it can be shown that $\mu_1(\Gamma) < 0$ if Γ approaches $\partial\Omega$ while $\mu_1(\Gamma) > 0$ if Γ is conveniently small (details are omitted for the sake of brevity). Therefore, $\lambda_1(\Gamma)$ is defined in this case depending upon the “size” of Γ.

Our next result deals with the continuity of the principal eigenvalue λ_1 with respect to variations in the flux region Γ. For the sake of simplicity, only the case $m(x) = \chi_\Gamma(x)$ will be considered. To this objective we are introducing a notion of perturbation which largely suffices for our purposes here (see Section 3). Let Γ_n be a sequence of smooth subdomains of $\partial\Omega$ and $\Gamma_0 \subset \partial\Omega$ a fixed subdomain. By

$$\lim \Gamma_n = \Gamma_0, \tag{2.13}$$

it is understood either one of the following two properties:

a) There exist sequences Γ_n', Γ_n'' of smooth subdomains such that $\Gamma_n' \subset \Gamma_n \cap \Gamma_0$ is increasing, $\Gamma_n'' \supset \Gamma_n \cup \Gamma_0$ is decreasing and $\lim \Gamma_n' = \lim \Gamma_n'' = \Gamma_0$.
b) There exist sequences Γ_n', Γ_n'', the former increasing, the latter a decreasing sequence, of smooth subdomains such that $\lim \Gamma_n' = \lim \Gamma_n'' = \Gamma_0$ and satisfying that for each n, the relations $\Gamma_n' \subset \Gamma_m$, $\Gamma_n'' \supset \Gamma_m$ hold for all $m \geqslant n$.

Notice that both conditions imply $\lim \Gamma_n = \Gamma_0$ in the set theory sense and that both definitions are coherent with monotone convergence.

Lemma 2.6. *Assume that Γ_n, Γ_0 are smooth subdomains of $\partial\Omega$ satisfying* (2.13)*. Then*:

i) $\lim \mu_1(\Gamma_n) = \mu_1(\Gamma_0)$.
ii) *$\lambda_1(\Gamma_n)$ is defined for large n provided $\mu_1(\Gamma_0) > 0$ and $\lim \lambda_1(\Gamma_n) = \lambda_1(\Gamma_0)$. Moreover, $\Phi_{1,n} \to \Phi_1$ in $H^1(\Omega)$ where $\Phi_{1,n}$ and Φ_1 stands for the normalized positive eigenfunctions associated to $\lambda_1(\Gamma_n)$ and $\lambda_1(\Gamma_0)$, respectively.*

Proof. The proofs of i) and ii) follow the same pattern. Thus, we are confining ourselves to show ii).

That $\lambda_1(\Gamma_n)$ is well defined for large n follows from i). On the other hand, no generality is lost if it is assumed in the sequel that $\lambda_1(\Gamma) > 0$ for all the involved subdomains $\Gamma \subset \partial\Omega$.

Setting $\lambda_n' = \lambda_1(\Gamma_n')$, it is clear from the definition (2.13) and Lemma 2.4 that it is enough to show that

$$\lim \lambda_n' = \lambda_1(\Gamma_0).$$

Fix $\lambda' = \lim \lambda_n'$ ($\lambda' \geqslant \lambda_1(\Gamma_0)$) and pick the sequence of normalized positive eigenfunctions Φ_n' associated to λ_n', and so $\int_{\partial\Omega} {\Phi_n'}^2 = 1$. Equality

$$\int_\Omega \left|\nabla \Phi_n'\right|^2 + q{\Phi_n'}^2 = \lambda_n' \int_{\Gamma_n} {\Phi_n'}^2$$

for all n implies that $\int_\Omega |\nabla \Phi_n'|^2 = O(1)$. Otherwise, set $\Phi_n' = t_n v_n$ with $t_n^2 = \int_\Omega |\nabla \Phi_n'|^2$. Then, passing to a subsequence, $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$ with $v = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla v|^2 + qv^2 \leqslant 0.$$

From (2.5) $\lambda_1^{\mathcal{D}}(q) > \sigma_1(\Gamma_0) > 0$ ($\lambda_1^{\mathcal{D}}(q)$ the first Dirichlet eigenvalue of $-\Delta + q$ in Ω), which says that $v = 0$. But now one has that $v_n \to v$ in $H^1(\Omega)$ which should imply that $\int_\Omega |\nabla v|^2 = 1$ which is impossible. Therefore, $\int_\Omega |\nabla \Phi_n'|^2$ is bounded, Φ_n' is bounded in $H^1(\Omega)$ and, modulus a subsequence, $\Phi_n' \to \Phi'$ with Φ' nonnegative together with $\int_{\partial\Omega} {\Phi'}^2 = 1$. By taking limits in the weak equations for Φ_n' we obtain that Φ' is a principal eigenfunction associated to λ'. Thus, the uniqueness of $\lambda_1(\Gamma_0)$ implies that $\lambda' = \lambda_1(\Gamma_0)$, as we wanted to show.

The proof of the convergence $\Phi_{1,n} \to \Phi_1$ in $H^1(\Omega)$ is already contained in the corresponding one of Theorem 5.6 below. □

Once the existence of the principal eigenvalue has been settled down, we need for subsequent use, a corresponding result of Fredholm alternative type. Such a result is next stated and a direct proof in a "variational guise" is also provided.

Theorem 2.7. *Suppose $m \in L^\infty(\Omega)$ is a nonnegative function supported on Γ so that condition (2.3) holds and let $f \in H^1(\Omega)^*$ (the dual space of $H^1(\Omega)$), $g \in L^2(\partial\Omega)$ be arbitrary. Then, the problem*

$$\begin{cases} -\Delta u + qu = f, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda_1 m u + g, & x \in \partial\Omega, \end{cases} \tag{2.14}$$

possesses a solution $u \in H^1(\Omega)$ if and only if the compatibility condition

$$\langle f, \Phi_1 \rangle + \int_{\partial\Omega} g\Phi_1 = 0 \tag{2.15}$$

holds, where $\Phi_1 \in H^1(\Omega)$ is any eigenfunction associated to λ_1 and $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H^1(\Omega)$ and its dual. Moreover, such a solution $u \in H^1(\Omega)$ is unique under the restriction

$$\int_{\partial\Omega} m\, u\Phi_1 = 0. \tag{2.16}$$

Proof. For simplicity in the notation we directly consider $m = \chi_\Gamma$, the characteristic function of Γ (the case m general is handled in the same way).

First of all, by a weak solution to (2.14) it is understood a function $u \in H^1(\Omega)$ such that the following equality holds

$$\int_\Omega \nabla u \nabla \psi + qu\psi = \lambda_1 \int_\Gamma u\psi + \langle f, \psi \rangle + \int_{\partial\Omega} g\psi, \quad \forall \psi \in H^1(\Omega). \tag{2.17}$$

Thus, if such a solution $u \in H^1(\Omega)$ exists then the necessity of (2.15) follows by choosing $\psi = \Phi_1$ in the above relation.

To show the sufficiency of (2.15) we first state the existence of a second eigenvalue $\lambda_2 > \lambda_1$ to (1.1). To this purpose, we introduce in $H^1(\Omega)$ the scalar product

$$[u, v] = \int_\Omega \nabla u \nabla v + quv + M \int_\Gamma uv, \tag{2.18}$$

where $M \geqslant 0$ is chosen so that $M + \lambda_1 > 0$. In fact, $[u,u] \geqslant (M + \lambda_1)\int_\Gamma u^2$ for all $u \in H^1(\Omega)$. Thus, thanks to condition (2.3) $[u,u] = 0$ implies $u = 0$. Moreover, by arguing as in the proof of inequality (2.21) below, it can be shown that (2.18) defines an equivalent norm in $H^1(\Omega)$.

To state the existence of λ_2 we now observe that eigenfunctions Φ to (1.1) associated to eigenvalues $\lambda \neq \lambda_1$ (and so $\lambda > \lambda_1$) must satisfy the orthogonality condition $\int_\Gamma \Phi\Phi_1 = 0$, which amounts to $[\Phi, \Phi_1] = 0$. Therefore, we study the quadratic functional $J(u) = \int_\Omega |\nabla u|^2 + qu^2$ on $\mathcal{M}_1 = \mathcal{M} \cap \{\Phi_1\}^\perp$, where $\mathcal{M}$ is the set defined in Theorem 2.1, $\{\Phi_1\}^\perp = \{u\colon\ [u, \Phi_1] = 0\}$ and where orthogonality "$\perp$" will be understood in the sequel with respect to $[\cdot,\cdot]$. Notice that

$$\{\Phi_1\}^\perp = \left\{ u \in H^1(\Omega)\colon \int_\Gamma u\Phi_1 = 0 \right\}.$$

By arguing as in the proof of Theorem 2.1, there exists a function $\Phi_2 \in \{\Phi_1\}^\perp$ such that

$$\lambda_2 := \inf_{u \in \{\Phi_1\}^\perp} \frac{\int |\nabla u|^2 + qu^2}{\int_\Gamma u^2} = \frac{\int |\nabla \Phi_2|^2 + q\Phi_2^2}{\int_\Gamma \Phi_2^2}. \tag{2.19}$$

Thus, Φ_2 satisfies

$$\int_\Omega \nabla\Phi_2 \nabla\psi + q\Phi_2\psi = \lambda_2 \int_\Gamma \Phi_2\psi, \quad \forall \psi \in \{\Phi_1\}^\perp. \tag{2.20}$$

To show that Φ_2 is an eigenfunction we need that the equality be true for all $\psi \in H^1(\Omega)$ (not merely for $\psi \in \{\Phi_1\}^\perp$). However, an arbitrary function $u \in H^1(\Omega)$ can be written as $u = t\Phi_1 + \psi$, with $\psi \in \{\Phi_1\}^\perp$, $t \in \mathbb{R}$. Now, since $\int_\Gamma \Phi_1\Phi_2 = 0$ we find $\int_\Omega \nabla\Phi_2\nabla\Phi_1 + q\Phi_2\Phi_1 = 0$.

Therefore, (2.20) holds for any $\psi \in H^1(\Omega)$. Hence, Φ_2 is a weak eigenfunction, λ_2 defines an eigenvalue and indeed constitutes the second eigenvalue to (1.1) (no other one lies between λ_1 and λ_2).

We are next showing the existence of a weak solution u^* to (2.14) provided that (2.15) holds. To this purpose consider the quadratic functional $F : \{\Phi_1\}^\perp \to \mathbb{R}$ defined as

$$F(u) = \frac{1}{2}\left(\int_\Omega \nabla u^2 + qu^2 - \lambda_1 \int_\Gamma u^2\right) - \langle f, u\rangle - \int_{\partial\Omega} gu.$$

Observe that

$$\int_\Omega \nabla u^2 + qu^2 - \lambda_1 \int_\Gamma u^2 \geqslant (\lambda_2 - \lambda_1) \int_\Gamma u^2, \quad \forall u \in \{\Phi_1\}^\perp.$$

We claim the existence of $C > 0$, no depending on $u \in \{\Phi_1\}^\perp$, such that

$$\int_\Omega \nabla u^2 + qu^2 - \lambda_1 \int_\Gamma u^2 \geqslant C\|u\|^2_{H^1(\Omega)}, \quad \forall u \in \{\Phi_1\}^\perp. \tag{2.21}$$

Assuming that the claim is true one obtains that the functional F is coercive on $\{\Phi_1\}^\perp$ which is a weakly closed part of $H^1(\Omega)$. This means that F achieves an absolute minimum at some $u^* \in \{\Phi_1\}^\perp$ and it implies, in particular, that the equation

$$\int_\Omega \nabla u^*\nabla\psi + qu^*\psi - \lambda_1 \int_\Gamma u^*\psi - \langle f, \psi\rangle - \int_{\partial\Omega} g\psi = 0 \tag{2.22}$$

holds provided $\psi \in \{\Phi_1\}^\perp$. To conclude that u^* is a weak solution we need replacing in such equation $\psi \in \{\Phi_1\}^\perp$ by $\psi \in H^1(\Omega)$. By writing $u \in H^1(\Omega)$ as $u = t\Phi_1 + \psi$ with $t \in \mathbb{R}$, $\psi \in \{\Phi_1\}^\perp$, we see that (2.22) is equivalent to

$$\int_\Omega \nabla u^*\nabla\Phi_1 + qu^*\Phi_1 - \lambda_1 \int_\Gamma u^*\Phi_1 - \langle f, \Phi_1\rangle - \int_{\partial\Omega} g\Phi_1 = 0. \tag{2.23}$$

Since $u^* \in \{\Phi\}^\perp$ such relation is equivalent to the compatibility condition (2.15). Therefore, u^* defines a weak solution to (2.14). Moreover, it is the unique solution to (2.14) in $\{\Phi_1\}^\perp$ since any other solution $\hat{u} \in \{\Phi_1\}^\perp$ exhibits the form $\hat{u} = u^* + t\Phi_1$, $t \in \mathbb{R}$. That $t = 0$ and so $\hat{u} = u^*$ follows from

$$0 = \int_\Gamma (\hat{u} - u^*)\Phi_1 = t \int_\Gamma \Phi_1^2.$$

To complete the proof we show now the claim. If a positive constant as C in (2.21) could not be found then a sequence $u_n \in \{\Phi_1\}^\perp$ would exist such that $\int_\Omega |\nabla v_n|^2 + q v_n^2 - \lambda_1 \int_\Gamma v_n^2 \to 0$ where $u_n = t_n v_n$ and $t_n = \|u_n\|_{H^1(\Omega)}$. By passing to a subsequence, $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$. Thanks to the inequality

$$\int_\Omega |\nabla v_n|^2 + q v_n^2 - \lambda_1 \int_\Gamma v_n^2 \geqslant (\lambda_2 - \lambda_1) \int_\Gamma v_n^2$$

we the achieve $v \in H^1_\Gamma(\Omega)$. Thus $\int_\Omega |\nabla v_n|^2 + q v_n^2 \to 0$. Taking 'lim–inf' we deduce $\int_\Omega |\nabla v|^2 + q v^2 \leqslant 0$. Being $\mu_1 > 0$ this implies that $v = 0$. But this entails that $v_n \to 0$ in $L^2(\Omega)$ what in turn says that $\int_\Omega |\nabla v_n|^2 \to 0$ and finally that $v_n \to 0$ in $H^1(\Omega)$. Due to the fact that $\|v_n\|_{H^1(\Omega)} = 1$ this is not possible, and the claim is proved. □

3. Tangential perturbations of Γ

In this section we introduce the notion of tangential deformation of the flux region $\Gamma \subset \partial\Omega$ which will be involved in the main perturbation results contained in next section. In addition, a further discussion on the differentiable structure of the boundary and other auxiliary calculus results on $\partial\Omega$ will be also included here.

We are considering a class C^2 vector field $V : \partial\Omega \to \mathbb{R}^N$ which is tangent to $\partial\Omega$ at every point. Recall that $\Omega \subset \mathbb{R}^N$ is assumed to be a class C^3 bounded domain. Hence, the field V can be extended as a smooth field on the whole $\mathbb{R}^N$ in such a way that $V \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$. For later use, it will be always assumed that such extension has been performed whenever the computations require it. On the other hand, a suitable extension of V which is parallel to $\partial\Omega$ near $\partial\Omega$ can always be constructed (see further details below).

Associated to the field V we set $h : \mathbb{R} \times \partial\Omega \to \partial\Omega$ the flow generated by V. Namely, for $x_0 \in \partial\Omega$, $x(t) = h(t, x_0)$ stands for the solution to the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = V(x), \\ x(0) = x_0. \end{cases}$$

We are using the same terminology $h = h(t, x)$, $h : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, to designate the flow of the extension of V to the whole $\mathbb{R}^N$. It is well known (see [9]) that $h \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$. In addition, the following properties hold true,

i) For every $t \in \mathbb{R}$ the mapping $h_t(x) := h(t, x)$ defines a class C^2 diffeomorphism in $\mathbb{R}^N$. The same is true when h_t is restricted both to Ω and $\partial\Omega$, i.e., when $h_t : \Omega \to \Omega$ and $h_t : \partial\Omega \to \partial\Omega$. Observe that both $\partial\Omega$ and Ω remain flow-invariant.

ii) $h_0(x) = x$ for all $x \in \mathbb{R}^N$. Moreover $(h_t)^{-1}(x) = h_{-t}(x)$ for all $x \in \mathbb{R}^N$.

iii) $D_t h(t, x) = V(h(t, x))$ and $D^2_{xt} h(t, x) = DV(h(t, x)) D_x h(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

For every $t \in \mathbb{R}$, we also introduce the composition map $h_t^* : C^2(\overline{\Omega}) \to C^2(\overline{\Omega})$ defined as

$$h_t^*(u)(x) = u\big(h(t,x)\big), \quad x \in \overline{\Omega}.$$

Of course, h_t^* is an isomorphism from $C^2(\overline{\Omega})$ onto itself.

On the other hand, and as usual in perturbation of domains theory ([15]), smooth tangent fields V are going to be used to define the perturbation of problem (1.1). This means that we are studying the smoothness of function $\lambda_1(\Gamma_t)$, where

$$\Gamma_t = \big\{h_t(x)\colon x \in \Gamma\big\},$$

and t is small.

As for the structure of $\partial\Omega$ it will be assumed that $\partial\Omega$ is endowed with a finite atlas $\{(g_i, U_i)\}_{1\leqslant i\leqslant M}$, $U_i \subset \mathbb{R}^{N-1}$ open, $g_i = g_i(s) \in C^3(U_i, \mathbb{R}^N)$, so that the restriction of the atlas to $\Gamma \cup \partial\Gamma$ constitutes an atlas for $\overline{\Gamma}$ as a manifold with boundary. Such atlas is chosen so that on every $g_i(U_i) \subset \partial\Omega$, the continuous outward normal field ν to Ω at $\partial\Omega$ can be expressed as

$$\nu\big(g_i(s)\big) = \frac{\partial_{s_1} g_i \wedge \cdots \wedge \partial_{s_{N-1}} g_i}{|\partial_{s_1} g_i \wedge \cdots \wedge \partial_{s_{N-1}} g_i|}.$$

As a matter of notation, for $N-1$ linearly independent vectors $v_1, \ldots, v_{N-1}$ of $\mathbb{R}^N$, $v_1 \wedge \cdots \wedge v_{N-1}$ will stands for the vector whose i-th coordinate is the adjoint of the element w_i in the matrix $W = \text{columns}(w, v_1, \ldots, v_{N-1})$, with $w = (w_1, \ldots, w_N)$. On the other hand, the collection of smooth functions $\{\zeta_i(x)\}_{1\leqslant i\leqslant M}$ will stands for a partition of unity associated to the atlas $\{(g_i, U_i)\}_{1\leqslant i\leqslant M}$.

The concept of tangential divergence (see [15]) is also involved in next section. For a nonnecessarily tangent smooth vector field V on $\partial\Omega$, its tangential divergence in $\partial\Omega$ is defined as the function $a \in C(\partial\Omega)$ such that

$$\int_{\partial\Omega} V(x)\nabla_{\partial\Omega}\psi(x)\,d\sigma = -\int_{\partial\Omega} a(x)\psi(x)\,d\sigma$$

for all $\psi \in C_0^1(\partial\Omega)$, where $\nabla_{\partial\Omega}\psi$ stands for the tangential component of $\nabla\psi$, i.e. $\nabla_{\partial\Omega}\psi(x) = \nabla\psi(x) - (\partial\psi/\partial\nu(x))\nu(x)$. We are denoting $a = \operatorname{div}_{\partial\Omega} V$.

The tangential divergence of V can be expressed in local coordinates (g, U) (subindex i is dropped for simplicity). In fact, a careful computation reveals that

$$\operatorname{div}_{\partial\Omega} V = \frac{1}{J}\big[|\partial_{s_1} V, \ldots, \partial_{s_{N-1}} g, \nu| + \cdots + |\partial_{s_1} g, \ldots, \partial_{s_{N-1}} V, \nu|\big] - \langle V, \nu\rangle H,$$

where $|\cdot|$ designates the determinant of the matrix whose columns are the vector enclosed between the bars, $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^N$ while $J = |\partial_{s_1} g \wedge \cdots \wedge \partial_{s_{N-1}} g|$. In addition, $H(x) = \operatorname{div}\nu(x)$ and

$$H = \frac{1}{J}\big[|\partial_{s_1}\nu, \ldots, \partial_{s_{N-1}} g, \nu| + \cdots + |\partial_{s_1} g, \ldots, \partial_{s_{N-1}}\nu, \nu|\big]. \tag{3.1}$$

It is well known that $H = \operatorname{div}\nu$ coincides – modulus orientation – with $(N-1)\mathcal{H}$ where $\mathcal{H}$ is the mean curvature of $\partial\Omega$ at x (see [26]). An alternative expression for $\operatorname{div}_{\partial\Omega} V$ can be found if one uses the so-called tubular coordinates around $\partial\Omega$. Namely, x is represented in a suitable neighborhood of $\partial\Omega$ as

$$x = z + t\nu(z), \tag{3.2}$$

with $z \in \partial\Omega$ and $t = d(x) = \operatorname{dist}(x, \partial\Omega)$ being $x \mapsto (t(x), z(x))$ a C^2 mapping near $\partial\Omega$. In that case, and by extending the normal ν so as to have

$$\nu\big(z + t\nu(z)\big) = \nu(z) \tag{3.3}$$

for $z \in \partial\Omega$ and $|t|$ small, the tangential divergence can be written as

$$\operatorname{div}_{\partial\Omega} V = \operatorname{div} V - \frac{\partial}{\partial\nu}\langle V, \nu\rangle - \langle V, \nu\rangle H. \tag{3.4}$$

Certainly, the curvature term can be omitted if V is tangent to $\partial\Omega$. Moreover, formula (3.4) can be further simplified by extending the field V in a neighborhood of $\partial\Omega$ such that

$$V(x) = V(z), \qquad x = z + t\nu(z), \quad \text{for } z \in \partial\Omega,\ |t| \text{ small}. \tag{3.5}$$

Observe that identity (3.5) can be employed to extend V outside $\partial\Omega$. Under this extension (3.4) reduces to

$$\operatorname{div}_{\partial\Omega} V = \operatorname{div} V, \tag{3.6}$$

when V is tangent to $\partial\Omega$. On the other hand, formula (3.1) can also be obtained by using the change to tubular coordinates (3.2).

4. Smoothness of λ_1 and a formula for its first variation

Our main objective in what follows will be to study the differentiable dependence of the principal eigenvalue λ_1 to (1.1), when the region Γ is perturbed by the flow $h_t(\cdot) = h(t, \cdot)$ associated to a tangent field V in $\partial\Omega$ (Section 3). In other words, the differentiability in t of the function

$$t \to \lambda_1(\Gamma_t),$$

$\Gamma_t = h_t(\Gamma)$ for $|t|$ small. In Theorem 4.1 we are proving the smoothness of such function while an explicit formula for its derivative is furnished in Theorem 4.2. Later in Theorem 5.1 an optimized version will be obtained.

Theorem 4.1. *Let Γ be a* smooth subdomain *of $\partial\Omega$ with boundary $\partial\Gamma$, while $V : \partial\Omega \to \mathbb{R}^N$ is a smooth tangent vector field to $\partial\Omega$ with associated flow $h : \mathbb{R} \times \partial\Omega \to \partial\Omega$. Define $\Gamma_t = \{y = h(t, x)\colon x \in \Gamma\}$, and consider the eigenvalue problem*

$$\begin{cases} -\Delta v + q(y)v = 0, & y \in \Omega, \\ \dfrac{\partial v}{\partial\nu} = \lambda\chi_{\Gamma_t}(y)v, & y \in \partial\Omega, \end{cases} \tag{4.1}$$

under the assumption that the principal eigenvalue $\mu_1(\Gamma)$ to (2.4) *is positive. Then, there exists $\varepsilon_0 > 0$ such that the following properties hold*:

i) *Problem* (4.1) *admits a principal eigenvalue $\lambda_1(t)$ for $t \in (-\varepsilon_0, \varepsilon_0)$. In addition, $\lambda_1(t)$ is a continuous function of t.*

ii) *If* $(\lambda_1(t), \Phi_1(t))$ *denotes the principal eigenvalue to* (4.1) *and corresponding positive eigenfunction normalized so that* $\int_{\Gamma_t} \Phi_1(t)^2 = 1$, *then the mapping*

$$t \to \big(\lambda_1(t), h_t^*\big(\Phi_1(t)\big)\big), \qquad h_t^*\big(\Phi_1(t)\big) = \Phi_1(t) \circ h_t$$

is smooth when regarded from $(-\varepsilon_0, \varepsilon_0)$ *and taking values in* $\mathbb{R} \times H^1(\Omega)$.

Proof. As a first observation, notice that $\Gamma_t \to \Gamma$ as $t \to 0$ in the sense of (2.13) since $\Gamma_t = h_t(\Gamma)$ and h_t is smooth in t. By using the continuous dependence of μ_1 on Γ (Lemma 2.6), condition $\mu_1(\Gamma) > 0$ implies the positivity of $\mu_1(\Gamma_t)$ for $|t| < \varepsilon_0$ and certain $\varepsilon_0 > 0$ small. Hence, Theorem 2.1 ensures us the existence of $\lambda_1(\Gamma_t)$, its continuity as a function of $t \in (-\varepsilon_0, \varepsilon_0)$ being provided by Lemma 2.6. Thus, i) is proved. For immediate use we fix the notation $\lambda_1(t)$ to denote the eigenvalue $\lambda_1(\Gamma_t)$ and $\Phi_1(t) \in H^1(\Omega)$ to name the normalized associated positive eigenfunction.

To show ii) we first set $X = H^1(\Omega)$ and $Y = (H^1(\Omega))^*$ (the dual space of X) and observe that if $v = v(y,t) \in X$ is an eigenfunction associated to an *arbitrary* eigenvalue λ of (4.1) then

$$\int_\Omega \big\{\nabla v \nabla\varphi + q(y) v\varphi\big\}\, dy = \lambda \int_{\Gamma_t} v\varphi\, d\sigma(y),$$

for all $\varphi \in X$. By performing the change $y = h_t(x)$, putting $u(x,t) = h_t^*(v)(x)$, $\psi(x) = h_t^*(\varphi)(x)$ we arrive at

$$\int_\Omega \big\{\mathcal{A}_t(x)(\nabla u, \nabla\psi) + h_t^*(q) u\psi\big\}\mathcal{C}_t(x)\, dx - \lambda \int_\Gamma u\psi \mathcal{D}_t(x)\, d\sigma(x) = 0, \tag{4.2}$$

for all $\psi \in X$, where $\mathcal{C}_t(x) = \det(Dh(t,x))$, $Dh(t,x) = (\partial_{x_j} h_i)_{1 \leqslant i,j \leqslant N}$, and

$$\mathcal{D}_t(x)\zeta_i(x) = \frac{|Dh(t,x)g_{s_1} \wedge \cdots \wedge Dh(t,x)g_{s_{N-1}}|}{|g_{s_1} \wedge \cdots \wedge g_{s_{N-1}}|},$$

at $x = g(s)$, $s \in U$, being $\{\zeta_i\}_{1 \leqslant i \leqslant M}$ the partition of the unity subordinated to the finite atlas $\{(g_i, U_i)\}$ which describes the differentiable structure of $\partial\Omega$ (Section 3). When writing the expression for $\mathcal{D}_t(x)$ and for the sake of brevity, we have dropped the subindex i in the chart (g_i, U_i). In addition, for $\xi, \eta \in \mathbb{R}^N$ the bilinear form $\mathcal{A}_t(x)(\xi, \eta)$ in (4.2) is defined through

$$\mathcal{A}_t(x)(\xi,\eta) = \xi\, Dh(t,x)^{-1}\big(Dh(t,x)^{-1}\big)^T \eta^T,$$

where for a vector $\eta = (\eta_1, \ldots, \eta_N)$ and an $N \times N$ matrix A, η^T and A^T mean the corresponding transposed objects.

In view of (4.2) we introduce now the mapping $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, $\mathcal{F} = \mathcal{F}(u, \lambda, t)$, $\mathcal{F} : X \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to Y \times \mathbb{R}$ given by

$$\big\langle \mathcal{F}_1(u,\lambda,t), \psi\big\rangle = \int_\Omega \big\{\mathcal{A}_t(x)(\nabla u, \nabla\psi) + h_t^*(q) u\psi\big\}\mathcal{C}_t(x)\, dx - \lambda \int_\Gamma u\psi \mathcal{D}_t(x)\, d\sigma$$

for $\psi \in X$ ($\langle\cdot,\cdot\rangle$ the duality pairing between X and Y) and

$$\mathcal{F}_2(u,\lambda,t) = \frac{1}{2}\bigg(\int_\Gamma u^2 \mathcal{D}(t,x)\, d\sigma - 1\bigg). \tag{4.3}$$

Then, the eigenvalues λ of (4.1) with associated eigenfunctions $v \in X$, normalized so that $\int_{\partial\Omega} v^2 = 1$, are characterized as the zeros $(u, \lambda, t) \in X \times \mathbb{R} \times \mathbb{R}$ to equation

$$\mathcal{F}(u, \lambda, t) = 0, \tag{4.4}$$

where $u = h_t^*(v)$. In other words, (4.4) constitutes the weak Lagrangian version of the perturbed problem (4.1).

Of course, our main purpose is solving with uniqueness equation (4.4) for (u, λ, t) close $(\Phi_1(0), \lambda_1(0), 0)$ in $X \times \mathbb{R} \times \mathbb{R}$.

It is clear that $\mathcal{F}$ is a C^1 mapping while $\mathcal{F}(\Phi_1(0), \lambda_1(0), 0) = 0$. On the other hand if $\mathcal{L} \in L(X \times \mathbb{R}, Y \times \mathbb{R})$ stands for the Frechet derivative of $\mathcal{F}$ with respect to (u, λ), evaluated at $(\Phi_1(0), \lambda_1(0), 0)$ and $(\hat{u}, \hat{\lambda}) \in X \times \mathbb{R}$ then

$$\mathcal{L}(\hat{u}, \hat{\lambda}) = \left(\int_\Omega \nabla\hat{u}\nabla\cdot + q\hat{u}\,\cdot - \hat{\lambda}\int_\Gamma \Phi_1(0)\cdot - \lambda_1(0)\int_\Gamma \hat{u}\,\cdot\,, \int_\Gamma \Phi_1(0)\hat{u} \right).$$

In this expression, the dot "$\cdot$" in the first component means the dual action of such component as an element of the dual space Y.

The operator $\mathcal{L}$ defines a topological isomorphism from $X \times \mathbb{R}$ onto $Y \times \mathbb{R}$. In fact, for $(f, \theta) \in Y \times \mathbb{R}$ given, the unique solution $(\hat{u}, \hat{\lambda})$ to equation

$$\mathcal{L}(\hat{u}, \hat{\lambda}) = (f, \theta),$$

is provided by the unique weak solution $\hat{u} \in X$ to the boundary value problem

$$\begin{cases} -\Delta\hat{u} + q\hat{u} = f, & x \in \Omega, \\ \dfrac{\partial\hat{u}}{\partial\nu} = \lambda_1(0)\chi_\Gamma(x)\hat{u} + \hat{\lambda}\chi_\Gamma\Phi_1(0), & x \in \partial\Omega, \end{cases} \tag{4.5}$$

which satisfies the extra condition

$$\int_\Gamma \Phi_1(0)\hat{u} = \theta. \tag{4.6}$$

Now, problem (4.5) admits a solution if and only if (Theorem 2.7)

$$\hat{\lambda} = -\langle f, \Phi_1(0)\rangle,$$

since $\int_\Gamma \Phi_1(0)^2 = 1$. This provides a unique value for $\hat{\lambda}$. For this value there exists a unique solution $u^* \in X$ to (4.5) such that $\int_\Gamma \Phi_1(0)u^* = 0$. All other remaining solutions $u \in X$ to (4.5) have the form $u = u^* + t\Phi_1(0)$. Thus, the choice $t = \theta$ furnishes the unique solution to (4.5)–(4.6).

Therefore, the Implicit Function Theorem, in its standard infinite-dimensional version (see [9]) permits us concluding the existence of $\varepsilon > 0$ (which, after possibly diminishing its value, we name again ε_0) and C^1 functions $\tilde{\lambda}_1(t)$, $u_1(t)$, the latter observed as taking values in X, such functions being defined in $(-\varepsilon_0, \varepsilon_0)$ and such that

$$\mathcal{F}\big(u_1(t), \tilde{\lambda}_1(t), t\big) = 0,$$

for $|t| < \varepsilon_0$, together with $(\tilde{\lambda}_1(0), u_1(0)) = (\lambda_1(0), \Phi_1(0))$.

On the other hand $(u, \lambda, t) = (h_t^*(\Phi_1(t)), \lambda_1(t), t)$ solves (4.4) for $|t| < \varepsilon_0$. Moreover, thanks to Lemma 2.6 we have that $(h_t^*(\Phi_1(t)), \lambda_1(t), t) \to (\Phi_1(0), \lambda_1(0), 0)$ in $X \times \mathbb{R} \times \mathbb{R}$ as $t \to 0$. Therefore, the uniqueness assertion of the Implicit Function Theorem enables us to conclude that

$$\big(\lambda_1(t), h_t^*\big(\Phi_1(t)\big)\big) = \big(\tilde{\lambda}_1(t), u_1(t)\big),$$

for $|t|$ small. Thus, the proof of point ii) is completed. $\square$

Our next task consists in obtaining an explicit formula for the first variation of λ_1 with respect to Γ. The natural way to do that is taking derivatives in Eq. (4.2). To this purpose we face the task of computing the surface integral

$$I = \int_{\partial\Omega} \frac{\partial \Phi_1(0)}{\partial \nu}\left(\Phi_1(0)\operatorname{div} V + 2\frac{\partial \Phi_1(0)}{\partial V}\right) d\sigma \tag{4.7}$$

($\partial/\partial V$ stands for the derivative in the direction of V). However, $\nabla\Phi_1(0)$ must exhibit some kind of discontinuity on $\partial\Gamma$ (Theorem 2.1) and hence the integrability of $\partial\Phi_1(0)/\partial V$ near $\partial\Gamma$ becomes unclear. Accordingly, we need to avoid the possible discontinuities of such function on $\partial\Gamma$. To this objective we are introducing some more notation. For $\delta > 0$ small we set

$$\Gamma_\delta^- = \big\{x \in \Gamma\colon \operatorname{dist}_{\partial\Omega}(x, \partial\Gamma) > \delta\big\}, \qquad \Gamma_\delta^+ = \big\{x \in \partial\Omega\colon \operatorname{dist}_{\partial\Omega}(x, \Gamma) > \delta\big\}$$

where for $A \subset \partial\Omega$ and $x \in \partial\Omega$, $\operatorname{dist}_{\partial\Omega}(x, A) = \inf_{y\in A} \operatorname{dist}_{\partial\Omega}(x, y)$, $\operatorname{dist}_{\partial\Omega}$ being the geodesic distance in $\partial\Omega$. Similarly, we put

$$U_\delta = \big\{x \in \partial\Omega\colon \operatorname{dist}_{\partial\Omega}(x, \partial\Gamma) < \delta\big\}, \qquad \Omega_\varepsilon = \big\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \varepsilon\big\},$$

where $\varepsilon > 0$ is small. Notice that $\partial\Omega = \overline{\Gamma_\delta^+} \cup \overline{\Gamma_\delta^-} \cup \overline{U_\delta}$ while $H_\varepsilon : \partial\Omega \to \partial\Omega_\varepsilon$ defined as $H_\varepsilon(z) = z - \varepsilon\nu(z)$ constitutes a C^2 diffeomorphism from $\partial\Omega$ onto $\partial\Omega_\varepsilon$. By means of H_ε, $\Gamma_\delta^\pm$ and U_δ are transported to $\partial\Omega_\varepsilon$ and we are setting

$$\Gamma_{\delta,\varepsilon}^\pm = H_\varepsilon\big(\Gamma_\delta^\pm\big), \qquad U_{\delta,\varepsilon} = H_\varepsilon(U_\delta).$$

In addition, $\partial\Omega_\varepsilon = \overline{\Gamma_{\delta,\varepsilon}^+} \cup \overline{\Gamma_{\delta,\varepsilon}^-} \cup \overline{U_{\delta,\varepsilon}}$.

Consider now the "displaced" surface integral

$$I_{\delta,\varepsilon} = \int_{U_{\delta,\varepsilon}} \frac{\partial \Phi_1(0)}{\partial \nu}\left(\Phi_1(0)\operatorname{div} V + 2\frac{\partial \Phi_1(0)}{\partial V}\right) d\sigma. \tag{4.8}$$

Since $\nabla\Phi_1(0)$ is discontinuous through $\partial\Gamma$, we cannot take the existence of the limit $\lim_{\varepsilon\to 0^+} I_{\delta,\varepsilon}$ for granted. Therefore, it is still less obvious that the iterated limit

$$I_0 := \lim_{\delta\to 0^+}\Big(\lim_{\varepsilon\to 0^+} I_{\delta,\varepsilon}\Big) \tag{4.9}$$

exists. Such an existence is provided in our next result. Its value is involved in the expression for the derivative of λ_1 with respect to t at $t = 0$ which is also furnished in the following statement.

Theorem 4.2. *Under the hypotheses of Theorem* 4.1*, let $\lambda_1 = \lambda_1(t)$ be the principal eigenvalue to* (4.1) *for $|t|$ small. Then the iterated limit I_0 in* (4.9) *exists. Moreover, the first variation of λ_1 with respect to t at $t = 0$ is given by*

$$\left.\frac{d\lambda_1}{dt}\right|_{t=0} = I_0 - \lambda_1(0) \int\limits_{\partial\Gamma} \Phi_1(0)^2 \langle V, \nu_{\partial\Gamma}\rangle \, d\sigma_{\partial\Gamma}, \tag{4.10}$$

where $\nu_{\partial\Gamma}$ stands for the outer unit normal to $\partial\Gamma$ relative to Γ, $d\sigma_{\partial\Gamma}$ is the volume element of $\partial\Gamma$ and $\Phi_1(0) \in H^1(\Omega)$ stands for the normalized positive eigenfunction associated to $\lambda_1(0)$.

Remark 4.3. If the pointwise limit as $\varepsilon \to 0^+$ could be permuted with the integral $I_{\delta,\varepsilon}$, i.e. if

$$\lim_{\varepsilon\to 0^+} I_{\delta,\varepsilon} = \lambda_1(0) \int\limits_{U_\delta} \chi_\Gamma \left[\Phi_1(0)^2 \operatorname{div} V + \frac{\partial}{\partial V}\big(\Phi_1(0)^2\big)\right], \tag{4.11}$$

then

$$\begin{aligned}\lim_{\varepsilon\to 0^+} I_{\delta,\varepsilon} &= \lambda_1(0) \int\limits_{U_\delta\cap\Gamma} \operatorname{div}\big(\Phi_1(0)^2 V\big) = \lambda_1(0) \int\limits_{U_\delta\cap\Gamma} \operatorname{div}_{\partial\Omega}\big(\Phi_1(0)^2 V\big) \\ &= \lambda_1(0)\left[\int\limits_{\partial\Gamma} \Phi_1(0)^2 \langle V, \nu_{\partial\Gamma}\rangle - \int\limits_{\{\operatorname{dist}_{\partial\Omega}(x,\partial\Gamma)=\delta\}} \Phi_1(0)^2 \langle V, \nu_{\partial\Gamma}\rangle\right].\end{aligned} \tag{4.12}$$

Since $\Phi_1(0) \in C^\alpha(\overline{\Omega})$ (Theorem 2.1) it is clear that the last expression in (4.12) goes to zero as $\delta \to 0^+$. Thus, the expected value for I_0 in (4.10) is just zero. However, the discontinuity of $\nabla\Phi_1(0)$ through $\partial\Gamma$ makes unclear that the ε limit can be permuted with the integral in (4.11). Nevertheless, by following a different approach we are showing in next section (Theorem 5.1) that $I_0 = 0$ provided q and Ω are sufficiently smooth.

Proof of Theorem 4.2. For convenience, we are using the notation of the proof of Theorem 4.1 and so we designate by

$$u(t,\cdot) = h_t^*\big(\Phi_1(t)\big) = \Phi_1(t) \circ h_t.$$

Differentiating with respect to t in (4.2) and setting $t = 0$ yields

$$\begin{aligned}&\int\limits_\Omega \{\nabla\dot{u}_0 \nabla\psi + q\dot{u}_0\psi\} - \lambda_1 \int\limits_\Gamma \dot{u}_0\psi + \int\limits_\Omega \left\{\dot{\mathcal{A}}(\nabla\Phi_1, \nabla\psi) + \frac{\partial q}{\partial V}\Phi_1\psi\right\} \\ &\quad + \int\limits_\Omega \{\nabla\Phi_1\nabla\psi + q\Phi_1\psi\}\dot{\mathcal{C}} - \dot{\lambda}_1 \int\limits_\Gamma \Phi_1\psi - \lambda_1 \int\limits_\Gamma \Phi_1\psi\dot{\mathcal{D}} = 0,\end{aligned} \tag{4.13}$$

for all $\psi \in H^1(\Omega)$, where $\dot{u}_0 = \partial_t u(0,x)$, $\Phi_1 = \Phi_1(0)$, $\lambda_1 = \lambda_1(0)$, $\dot{\lambda}_1 = \dot{\lambda}_1(0) = d\lambda_1/dt$ at $t = 0$. In addition $\dot{\mathcal{A}}(x) = \partial_t\mathcal{A}(0,x)$, $\dot{\mathcal{C}}(x) = \partial_t\mathcal{C}(0,x)$ and $\dot{\mathcal{D}}(x) = \partial_t\mathcal{D}(0,x)$. In the latter three relations, the notation $\mathcal{A}(t,x) = \mathcal{A}_t(x)$, $\mathcal{C}(t,x) = \mathcal{C}_t(x)$ and $\mathcal{D}(t,x) = \mathcal{D}_t(x)$ has been employed.

From the definitions of $\mathcal{A}, \mathcal{C}$ and $\mathcal{D}$ (Theorem 4.1) it can be checked that

$$\dot{\mathcal{A}}(x)(\xi,\eta) = -\xi\big(DV + DV^T\big)\eta^T, \quad \xi,\eta \in \mathbb{R}^N,$$

while $\dot{\mathcal{C}} = \operatorname{div} V$. On the other hand, a direct computation shows that

$$\dot{\mathcal{D}}(x) = \frac{1}{J(s)}\big[|\partial_{s_1} V, \dots, \partial_{s_{N-1}} g, \nu| + \dots + |\partial_{s_1} g, \dots, \partial_{s_{N-1}} V, \nu|\big],$$

where such expression has been evaluated at the image $g_i(U_i)$ of a chart (g_i, U_i) of $\partial\Omega$ (i has been dropped for simplicity). Taking into account the first identity for $\operatorname{div}_{\partial\Omega} V$ in Section 3 together with the fact that V is a tangent field on $\partial\Omega$ we can write

$$\dot{\mathcal{D}}(x) = \operatorname{div}_{\partial\Omega} V(x), \quad x \in \partial\Omega.$$

Furthermore, by employing the parallel extension (3.5) of V near $\partial\Omega$ and (3.6) we conclude that $\dot{\mathcal{D}}(x) = \operatorname{div} V(x)$.

By substituting the values of $\dot{\mathcal{A}}, \dot{\mathcal{C}}, \dot{\mathcal{D}}$ in (4.13) and taking into account the identity,

$$\begin{aligned}\int_\Omega &\big\{\nabla\Phi_1(\operatorname{div} V)\nabla\psi + q\Phi_1(\operatorname{div} V)\psi\big\} \\ &= \lambda_1 \int_\Gamma (\operatorname{div} V)\Phi_1\psi - \int_\Omega \psi\nabla\Phi_1\nabla(\operatorname{div} V),\end{aligned}$$

which follows from the weak equation for $\Phi_1 = \Phi_1(0)$, we deduce that

$$\begin{aligned}\int_\Omega \{\nabla\dot{u}_0\nabla\psi + q\dot{u}_0\psi\} - \lambda_1\int_\Gamma \dot{u}_0\psi = \int_\Omega &\left\{\nabla\Phi_1\big(DV + DV^T\big)\nabla\psi^T - \frac{\partial q}{\partial V}\Phi_1\psi\right\} \\ &+ \int_\Omega \psi\nabla\Phi_1\nabla(\operatorname{div} V) + \dot{\lambda}_1\int_\Gamma \Phi_1\psi\end{aligned}$$

holds for all $\psi \in H^1(\Omega)$. But this means that $w = \dot{u}_0$ defines a weak solution to

$$\begin{cases} -\Delta w + qw = f, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = \lambda_1(0)\chi_\Gamma w + g, & x \in \partial\Omega, \end{cases} \tag{4.14}$$

with $f \in (H^1(\Omega))^*$ defined as

$$\langle f, \psi\rangle = \int_\Omega \left\{\nabla\Phi_1\big(DV + DV^T\big)\nabla\psi^T - \frac{\partial q}{\partial V}\Phi_1\psi + \psi\nabla\Phi_1\nabla(\operatorname{div} V)\right\},$$

and $g = \dot{\lambda}_1\chi_\Gamma\Phi_1(0)$ (recall that $\Phi_1 = \Phi_1(0)$).

Therefore, compatibility condition (2.15) yields

$$\dot{\lambda}_1 + \int_\Omega \nabla\Phi_1\big(DV + DV^T\big)\nabla\Phi_1^T + \int_\Omega \Phi_1\nabla\Phi_1\nabla(\operatorname{div} V) - \int_\Omega \frac{\partial q}{\partial V}\Phi_1^2 = 0, \tag{4.15}$$

which gives an explicit expression for $\dot{\lambda}_1 = \dot{\lambda}_1(0)$.

Differentiating equation $\mathcal{F}_2(u(\cdot,t),\lambda_1(t),0)=0$ (Theorem 4.1) at $t=0$, we obtain

$$\int_\Gamma \dot{u}_0\Phi_1 = -\frac{1}{2}\int_\Gamma \Phi_1\dot{\mathcal{D}}.$$

This is just the normalization condition that permit us solving problem (4.14) for $\dot{u}_0$ with uniqueness (see Theorem 2.7).

We proceed next to clear out the expression for $\dot{\lambda}_1=\dot{\lambda}_1(0)$ in (4.15). By setting

$$A_\varepsilon = \int_{\Omega_\varepsilon} \nabla\Phi_1\big(DV+DV^T\big)\nabla\Phi_1^T, \qquad B_\varepsilon = \int_{\Omega_\varepsilon} \Phi_1\nabla\Phi_1\nabla(\operatorname{div} V),$$

it is clear that

$$\lim_{\varepsilon\to0^+} A_\varepsilon = \int_\Omega \nabla\Phi_1\big(DV+DV^T\big)\nabla\Phi_1^T, \qquad \lim_{\varepsilon\to0^+} B_\varepsilon = \int_\Omega \big(\nabla\Phi_1\nabla(\operatorname{div} V)\big)\Phi_1,$$

and so,

$$-\dot{\lambda}_1 = \lim_{\varepsilon\to0^+}[A_\varepsilon+B_\varepsilon] - \int_{\partial\Omega} \Phi_1^2\frac{\partial q}{\partial V}. \tag{4.16}$$

On the other hand, integration by parts gives

$$A_\varepsilon = \int_{\Omega_\varepsilon}\left\{|\nabla\Phi_1|^2\operatorname{div} V - 2\frac{\partial\Phi_1}{\partial V}\Delta\Phi_1\right\} + 2\int_{\partial\Omega_\varepsilon}\frac{\partial\Phi_1}{\partial\nu}\frac{\partial\Phi_1}{\partial V} - \int_{\partial\Omega_\varepsilon}|\nabla\Phi_1|^2\langle V,\nu\rangle,$$

where the last integral vanishes for ε small due to $\langle V,\nu\rangle=0$ near $\partial\Omega$ (see (3.5)). In addition

$$B_\varepsilon = -\int_{\Omega_\varepsilon}\big\{\Phi_1\Delta\Phi_1+|\nabla\Phi_1|^2\big\}\operatorname{div} V + \int_{\partial\Omega_\varepsilon}\Phi_1\frac{\partial\Phi_1}{\partial\nu}\operatorname{div} V.$$

Thus, by choosing $\delta>0$ small we obtain

$$\begin{aligned}
A_\varepsilon+B_\varepsilon &= -\int_{\Omega_\varepsilon} q\operatorname{div}\big(\Phi_1^2V\big) + \int_{\partial\Omega_\varepsilon}\left(\Phi_1\operatorname{div} V + 2\frac{\partial\Phi_1}{\partial V}\right)\frac{\partial\Phi_1}{\partial\nu}\\
&= \int_{\Omega_\varepsilon}\Phi_1^2\frac{\partial q}{\partial V} + I_{\delta,\varepsilon} + \left\{\int_{\Gamma^+_{\delta,\varepsilon}} + \int_{\Gamma^-_{\delta,\varepsilon}}\right\}\left(\Phi_1\operatorname{div} V + 2\frac{\partial\Phi_1}{\partial V}\right)\frac{\partial\Phi_1}{\partial\nu}.
\end{aligned}$$

On the other hand,

$$\lim_{\varepsilon\to 0^+}\left\{\int\limits_{\Gamma^+_{\delta,\varepsilon}}+\int\limits_{\Gamma^-_{\delta,\varepsilon}}\right\}\left(\Phi_1\operatorname{div}V+2\frac{\partial\Phi_1}{\partial V}\right)\frac{\partial\Phi_1}{\partial\nu}$$
$$=\lambda_1\int\limits_{\Gamma^-_\delta}\operatorname{div}\left(\Phi_1^2V\right)=\lambda_1\int\limits_{\Gamma^-_\delta}\operatorname{div}_{\partial\Omega}\left(\Phi_1^2V\right)=\int\limits_{\partial\Gamma^-_\delta}\Phi_1^2\langle V,\nu_{\partial\Gamma^-_\delta}\rangle,$$

where $\nu_{\partial\Gamma^-_\delta}$ stands for the outward unit normal to Γ^-_δ at $\partial\Gamma^-_\delta$ and the divergence theorem for manifolds with boundary has been employed ([7]). Therefore

$$\lim_{\varepsilon\to 0^+}[A_\varepsilon+B_\varepsilon]=\int\limits_{\partial\Omega}\Phi_1^2\frac{\partial q}{\partial V}+\int\limits_{\partial\Gamma^-_\delta}\Phi_1^2\langle V,\nu_{\partial\Gamma^-_\delta}\rangle+\lim_{\varepsilon\to 0^+}I_{\delta,\varepsilon},$$

where the existence of the last limit is directly furnished by the equality. By substituting in (4.16) and restoring the notation $\dot\lambda_1(0)$ and $\Phi_1(0)$ instead of $\dot\lambda_1$ and Φ_1 we get

$$-\dot\lambda_1(0)=\lim_{\varepsilon\to 0^+}I_{\delta,\varepsilon}+\int\limits_{\partial\Gamma^-_\delta}\Phi_1(0)^2\langle V,\nu_{\partial\Gamma^-_\delta}\rangle.$$

Finally, by observing that $\Phi_1(0)\in C^\alpha(\overline{\Omega})$ and taking limits in the last expression as $\delta\to 0^+$ we obtain both the existence of the iterated limit (4.9) together with formula (4.10) for $\dot\lambda_1(0)$. This concludes the proof. □

5. The first variation of λ_1 on smooth domains

The objective of this section is showing that formula (4.10) for the derivative of the principal eigenvalue $\lambda=\lambda_1(t)$ to problem (4.1) can be improved by removing the term I_0.

Such a formula is obtained in next result under extra smoothness on both q and Ω. We proceed in this way by the sake of simplicity since such requirement may be considerably weakened. We are assuming in addition that $-\Delta+q$ is invertible under Neumann conditions. This is a mere technical assumption and may be removed (see Remark 5.13).

Theorem 5.1. *Under the hypotheses of Theorem* 4.1 *assume in addition that Ω is C^∞, $q\in C^\infty(\overline{\Omega})$ and that none of the Neumann eigenvalues of $-\Delta+q$ in Ω vanishes. Then, the derivative of the principal eigenvalue $\lambda_1(t)$ to* (4.1) *is given by the expression*

$$\left.\frac{d\lambda_1}{dt}\right|_{t=0}=-\lambda_1(0)\int\limits_{\partial\Gamma}\Phi_1(0)^2\langle V,\nu_{\partial\Gamma}\rangle\,d\sigma_{\partial\Gamma},\tag{5.1}$$

where $\nu_{\partial\Gamma}$ stands for the outer unit normal to $\partial\Gamma$ relative to Γ, $d\sigma_{\partial\Gamma}$ is the volume element of $\partial\Gamma$ and $\Phi_1(0)$ stands for the normalized positive eigenfunction associated to $\lambda_1(0)$.

Remark 5.2. Notice that $\lambda_1^{\mathcal{N}}(q)>0$ both implies (2.3) and the invertibility of $-\Delta+q$ under Neumann conditions. On the other hand, extra smoothness on q and Ω is only needed in the proof of Theorem 5.8 below.

To show Theorem 5.1 we proceed by successive steps. Our first result gives a derivative's formula for a "regularized" version of (4.1).

Lemma 5.3. *Suppose $\Gamma \subset \partial\Omega$ and $V = V(x)$ satisfy the hypotheses of Theorem* 4.1 *and choose $m \in C^2(\partial\Omega)$ a nonnegative function supported in Γ. Then, problem*

$$\begin{cases} -\Delta v + q(y)v = 0, & y \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = \lambda m(y,t)v, & y \in \partial\Omega, \end{cases} \tag{5.2}$$

with $m(y,t) = h^*_{-t}(m)(y) = m(h(-t,y))$ possesses the following properties:

i) *There exits ε_0 such that* (5.2) *has a principal eigenvalue $\lambda = \lambda_1(t)$ for $|t| < \varepsilon_0$ being $\lambda_1(t)$ a C^1 function in $(-\varepsilon_0, \varepsilon_0)$.*
ii) *The derivative $\dot\lambda_1$ of λ_1 at $t = 0$ can be expressed as*

$$\dot\lambda_1(0) = -\lambda_1(0) \int\limits_{\partial\Omega} m \operatorname{div}\big(\Phi_1(0)^2 V\big)\, d\sigma, \tag{5.3}$$

where $(\lambda_1(0), \Phi_1(0))$ is the principal eigenpair corresponding to $t = 0$ and $\Phi_1(0)$ is positive and normalized according to $\int_{\partial\Omega} m\Phi_1(0)^2\, d\sigma = 1$.

Remark 5.4. Observe that $\chi_{\Gamma_t}(y) = h^*_{-t}(\chi_\Gamma)(y)$ for all $y \in \partial\Omega$ and $t \in \mathbb{R}$. This shows the coincidence between (4.1) and (5.2) when $m = \chi_\Gamma$.

Proof of Lemma 5.3. The proof of existence and smoothness of $\lambda_1(t)$ is that one of Theorem 4.1 but in a better scenario: χ_Γ is replaced with a smooth function $m \in C^2(\partial\Omega)$. That is why, and thanks to Theorem 2.1-iii), that the normalized principal eigenfunction $\Phi_1(0) \in C^2(\overline{\Omega})$.

Now, by keeping the notation of Theorem 4.2 and employing the regularity of $\Phi_1 = \Phi_1(0)$ up to the boundary $\partial\Omega$ we achieve that $A_\varepsilon \to A$, $B_\varepsilon \to B$ with

$$A = \int\limits_\Omega \left\{ |\nabla\Phi_1|^2 \operatorname{div} V - 2\frac{\partial\Phi_1}{\partial V}\Delta\Phi_1 \right\} + 2\int\limits_{\partial\Omega} \frac{\partial\Phi_1}{\partial\nu}\frac{\partial\Phi_1}{\partial V} - \int\limits_{\partial\Omega} |\nabla\Phi_1|^2 \langle V, \nu\rangle,$$

and

$$B = -\int\limits_\Omega \left\{ \Phi_1\Delta\Phi_1 + |\nabla\Phi_1|^2 \right\} \operatorname{div} V + \int\limits_{\partial\Omega} \Phi_1 \frac{\partial\Phi_1}{\partial\nu} \operatorname{div} V.$$

Thus,

$$-\dot\lambda_1(0) = A + B - \int\limits_\Omega \frac{\partial q}{\partial V}\Phi_1(0)^2 = I,$$

where I is the integral in (4.7). Being $\Phi_1(0)$ smooth up to $\partial\Omega$ we obtain

$$I = \lambda_1(0) \int\limits_{\partial\Omega} m \operatorname{div}\big(\Phi_1(0)^2 V\big)\, d\sigma,$$

and the proof is concluded. $\square$

Our next result states that function $m_0(x) = \chi_\Gamma(x)$ can be suitably approximated by smooth functions defined on $\partial\Omega$. Its proof involves the use of a partition of unity and standard regularization and so is omitted.

Lemma 5.5. *Let Γ be a smooth subdomain of $\partial\Omega$ and set $m_0 = \chi_\Gamma$. Then there exists a family of nonnegative functions $m_\varepsilon \in C^\infty(\partial\Omega)$, $0 < \varepsilon < \varepsilon_1$ such that*

i) *$m_\varepsilon \to m_0$ in $L^q(\partial\Omega)$ for all $1 \leqslant q < \infty$.*
ii) *$\|m_\varepsilon\|_{\infty,\partial\Omega} \leqslant K$ for certain $K > 0$.*
iii) *$\Gamma_\varepsilon := \operatorname{supp} m_\varepsilon \subset \{x \in \partial\Omega\colon \operatorname{dist}_{\partial\Omega}(x,\Gamma) < \delta\}$ where $\delta = \delta(\varepsilon)$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Assume now that Γ satisfies the hypotheses of Theorem 4.1, in particular condition (2.3),

$$\mu_1(\Gamma) > 0.$$

It follows from Lemma 2.6 that $\mu_1(\Gamma_\varepsilon) > 0$ for ε small, say $0 < \varepsilon < \varepsilon_0$, where $\Gamma_\varepsilon := \operatorname{supp} m_\varepsilon$ (Lemma 5.5). Therefore, Theorem 2.1 provides the existence of a unique principal eigenpair $(\lambda, u) = (\lambda_{1,\varepsilon}, \Phi_{1,\varepsilon})$ to

$$\begin{cases} -\Delta u + q(x)u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda m_\varepsilon(x) u, & x \in \partial\Omega, \end{cases} \tag{5.4}$$

with $\Phi_{1,\varepsilon}$ the positive eigenfunction normalized so as $\int_{\partial\Omega} m_\varepsilon \Phi_{1,\varepsilon}^2 = 1$.

We are now in position to get a limit expression for the derivative of the principal eigenvalue $\lambda_1(t)$ to (4.1).

Theorem 5.6. *Under the assumptions of Theorem 4.1 let $\lambda = \lambda_1(t)$ the principal eigenvalue to (4.1). Then $\dot\lambda_1(0) = \frac{d\lambda_1}{dt}_{|t=0}$ satisfies*

$$\dot\lambda_1(0) = -\lambda_1(0) \lim_{\varepsilon\to 0} \int\limits_{\partial\Omega} m_\varepsilon \operatorname{div}\bigl(\Phi_{1,\varepsilon}^2 V\bigr)\, d\sigma. \tag{5.5}$$

Proof of Theorem 5.6 relies upon the following generalization of Theorem 4.1.

Lemma 5.7. *Assume that $\Gamma \subset \partial\Omega$ and $V = V(x)$ fulfill the requirements of Theorem 4.1 and fix $q > N-1$. For $t \in \mathbb{R}$, $m \in L^q(\partial\Omega)$ consider the problem*

$$\begin{cases} -\Delta v + q(y)v = 0, & y \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = \lambda m(y,t) v, & y \in \partial\Omega, \end{cases} \tag{5.6}$$

where $m(y,t) = h^*_{-t}(m)(y) = m(h(-t,y))$ and set $m_0 = \chi_\Gamma(x)$. Then there exist positive numbers $\varepsilon_0, \delta, \eta$ and class C^1 mappings $\lambda = \lambda(m,t)$, $u = u(m,t)$, $\lambda : B(m_0,\delta) \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, $u : B(m_0,\delta) \times (-\varepsilon_0,\varepsilon_0) \to H^1(\Omega)$, with $B(m_0,\delta) = \{m \in L^q(\partial\Omega)\colon \|m - m_0\|_{L^q(\partial\Omega)} < \delta\}$, such that,

i) *$(\lambda, v) = (\lambda(m,t), h^*_{-t}(u(m,t)))$ constitutes an eigenpair to (5.6) for all $m \in B(m_0,\delta)$, $|t| < \varepsilon_0$ satisfying*

$$\int\limits_{\partial\Omega} m(\cdot,t) v^2 = 1. \tag{5.7}$$

Moreover, $(\lambda(m,t), h^*_{-t}(u(m,t))) = (\lambda_1(0), \Phi_1(0))$ *at* $m = m_0$, $t = 0$, *where* $(\lambda_1(0), \Phi_1(0))$ *stands for the principal normalized eigenpair to* (1.1)*.*

ii) *If* (λ, v) *is an eigenpair to* (5.6) *with* $|\lambda - \lambda_1(0)| < \eta$, $\|v - \Phi_1(0)\|_{H^1(\Omega)} < \eta$ *and* v *satisfies in addition* (5.7) *then, necessarily*

$$(\lambda, v) = \big(\lambda(m,t), v(m,t)\big),$$

for a certain $(m,t) \in B(m_0, \delta) \times (-\varepsilon_0, \varepsilon_0)$ *where* $v(m,t) = h^*_{-t}(u(m,t))$*.*

iii) *If* $\dot\lambda_1(t)$ *stands for the derivative of the principal eigenvalue* $\lambda = \lambda_1(t)$ *to* (4.1) *then*

$$\dot\lambda_1(0) = \lim_{(m,t)\to(m_0,0)} \frac{\partial\lambda}{\partial t}(m,t). \tag{5.8}$$

Proof. Following the program of the proof of Theorem 4.1 (the notation used there is kept) we set $Z = L^q(\partial\Omega)$ with $q > N-1$, and consider the mapping $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, $\mathcal{F} : X \times Z \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to Y \times \mathbb{R}$, where

$$\big\langle \mathcal{F}_1(u,m,\lambda,t), \psi\big\rangle = \int_\Omega \big[\mathcal{A}_t(x)(\nabla u, \nabla\psi) + h_t^*(q) u\psi\big]\mathcal{C}_t(x)\,dx - \lambda \int_{\partial\Omega} m u \psi \mathcal{D}_t(x)\,d\sigma$$

for $\psi \in X$ and $\mathcal{F}_2(u,m,\lambda,t) = \frac{1}{2}(\int_{\partial\Omega} m u^2 \mathcal{D}(t,x)\,d\sigma - 1)$.

Then, the eigenvalues λ of (5.6) with associated eigenfunctions $v \in X$, normalized so that $\int_{\partial\Omega} m v^2 = 1$, are characterized as the zeros $(u,m,\lambda,t) \in X \times Z \times \mathbb{R} \times \mathbb{R}$ of the equation

$$\mathcal{F}(u,m,\lambda,t) = 0, \tag{5.9}$$

where $u = h_t^*(v)$.

We now observe that the inclusion $H^1(\Omega) \hookrightarrow L^p(\partial\Omega)$ is continuous for all $p \geqslant 1$ if $N = 2$, and for $1 \leqslant p \leqslant p^*_{\partial\Omega}$, $p^*_{\partial\Omega} = 2(N-1)/(N-2)$, if $N \geqslant 3$ ([16]). Thus, mapping $\mathcal{F} = \mathcal{F}(u,m,\lambda,t)$ is linear continuous with respect to $m \in Z$ provided $q > N-1$.

As shown in Theorem 4.1, Implicit Function Theorem can be employed to solve with uniqueness equation (5.9) near $(u,m,\lambda,t) = (\Phi_1(0), m_0, \lambda_1(0), 0)$. This yields assertions i), ii), while relation (5.8) is nothing else but the continuity of $\frac{\partial\lambda}{\partial t}(m,t)$ at $(m,t) = (m_0, 0)$. □

Proof of Theorem 5.6. Let m_ε be the regularizing sequence introduced in Lemma 5.5. If $(\lambda, u) = (\lambda_{1,\varepsilon}, \Phi_{1,\varepsilon})$ stands for the normalized principal eigenpair to (5.4), we claim that $\lambda_{1,\varepsilon} \to \lambda_1(0)$ and that $\Phi_{1,\varepsilon} \to \Phi_1(0)$ in $H^1(\Omega)$.

By assuming the claim, we conclude in view of ii) of Lemma 5.7 that $\lambda(m_\varepsilon, 0) = \lambda_{1,\varepsilon}$ for ε small. Since (Lemma 5.3)

$$\frac{\partial\lambda}{\partial t}(m_\varepsilon, 0) = -\lambda_{1,\varepsilon} \int_{\partial\Omega} m_\varepsilon \operatorname{div}\big(\Phi^2_{1,\varepsilon} V\big)\,d\sigma,$$

then (5.5) follows from (5.8) by setting $m = m_\varepsilon$, $t = 0$ and making $\varepsilon \to 0$.

For the sake of completeness we next give a direct proof of the claim. Assuming that both $\lambda_{1,\varepsilon}$ and $\|\Phi_{1,\varepsilon}\|_{H^1(\Omega)}$ are bounded we achieve the assertion. In fact, taking $\varepsilon_n \to 0$, setting $u_n = \Phi_{1,\varepsilon_n}$, $\lambda_n = \lambda_{1,\varepsilon_n}$, $m_n = m_{\varepsilon_n}$ and passing through a subsequence we see that $u_n \rightharpoonup u_0$ weakly in $H^1(\Omega)$,

$u_n \to u_0$ both strongly in $L^2(\Omega)$ and in $L^p(\partial\Omega)$, $p < p^*_{\partial\Omega}$, for in this case the embedding $H^1(\Omega) \hookrightarrow L^p(\partial\Omega)$ is compact ([16]). Thus $m_n u_n \varphi \to m_0 u_0 \varphi$ in $L^1(\partial\Omega)$ for all $\varphi \in H^1(\Omega)$. By passing to limits in

$$\int_\Omega \nabla u_n \nabla\varphi + q u_n \varphi = \lambda_n \int_{\partial\Omega} m_n u_n \varphi,$$

we arrive to

$$\int_\Omega \nabla u_0 \nabla\varphi + q u_0 \varphi = \lambda' \int_{\partial\Omega} m_0 u_0 \varphi$$

with λ' a limit point of λ_n. Since $u_0 \geqslant 0$, $\int_{\partial\Omega} m_0 u_0^2 = 1$ then $(\lambda', u_0) = (\lambda_1(0), \Phi_1(0))$. This shows that $\lambda_{1,\varepsilon} \to \lambda_1(0)$ and that the whole $\Phi_{1,\varepsilon} \rightharpoonup \Phi_1(0)$ weakly in $H^1(\Omega)$. The convergence in $H^1(\Omega)$ follows from the fact that $\Phi_{1,\varepsilon} \to \Phi_1(0)$ in $L^2(\Omega)$ together with

$$\int_\Omega \left|\nabla\Phi_1(0)\right|^2 = \lambda_1(0) - \int_\Omega q\Phi_1(0)^2 = \lim_{\varepsilon\to 0} \int_\Omega |\nabla\Phi_{1,\varepsilon}|^2.$$

We now show the boundedness of $\lambda_{1,\varepsilon}$. First we have

$$\lambda_{1,\varepsilon} \int_{\partial\Omega} m_\varepsilon \Phi_1^2 \leqslant \int_\Omega |\nabla\Phi_1|^2 + q\Phi_1^2 = \lambda_1(0),$$

and so $\overline{\lim}\,\lambda_{1,\varepsilon} \leqslant \lambda_1(0)$. Second, $\lambda_{1,\varepsilon}$ is bounded below, otherwise $\lambda_n = \lambda_{1,\varepsilon_n} \to -\infty$ with $\varepsilon_n \to 0$. Using the previous notation and putting $u_n = |\lambda_n|^{1/2} v_n$ we get

$$\int_\Omega |\nabla v_n|^2 + q v_n^2 = -1.$$

If $\|v_n\|_{H^1(\Omega)}$ is bounded, $v_n \rightharpoonup v_0$ weakly in $H^1(\Omega)$ with $v_0 = 0$ on Γ and

$$\int_\Omega |\nabla v_0|^2 + q v_0^2 \leqslant -1.$$

This contradicts (2.3) and so $\int_\Omega v_n^2 \to \infty$. In this case, setting $s_n = \|v_n\|_{L^2(\Omega)}$ and $v_n = s_n w_n$ we find $\int_\Omega |\nabla w_n|^2 + q w_n^2 = -s_n^{-2}$. By extracting from w_n a weakly converging subsequence with limit w_0 in $H^1(\Omega)$, we get $w_0 \neq 0$ with

$$\int_\Omega |\nabla w_0|^2 + q w_0^2 \leqslant 0.$$

This contradicts again (2.3). Therefore, $\lambda_{1,\varepsilon}$ keeps bounded.

Finally, a similar argument proves the boundedness of $\Phi_{1,\varepsilon}$ in $H^1(\Omega)$. $\square$

Next statement provides the last step to show Theorem 5.1. This is just the unique part of the proof where the extra smoothness of q and Ω is involved.

Theorem 5.8. *Suppose the assumptions of Theorem* 5.1 *hold and let $\Phi_{1,\varepsilon}$ be the principal positive eigenfunction to* (5.4) *satisfying $\int_{\partial\Omega} m_\varepsilon \Phi_{1,\varepsilon}^2 = 1$. Then, both $\Phi_{1,\varepsilon}$ and $\Phi_1(0)$ belong to $H^1(\partial\Omega)$. Moreover,*

$$\Phi_{1,\varepsilon} \to \Phi_1(0) \quad \text{in } H^1(\partial\Omega). \tag{5.10}$$

In order to proceed further we first need to introduce some definitions taken from [18]. Schwartz's notation ∂^α for derivatives of order $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha \in \{\mathbb{Z}^+\}^N$, $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, is followed below.

Definition 5.9. (See [18, p. 183].) For $s = 0, 1, \dots,$ let

$$\Xi^s(\Omega) = \left\{u \in L^2(\Omega)\colon \bar{d}^{|\alpha|}\partial^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leqslant s\right\},$$

with norm $\|u\|_{\Xi^s(\Omega)} = \sum_{|\alpha|\leqslant s} \|\bar{d}^{\,|\alpha|}\partial^\alpha u\|_{L^2(\Omega)}$, where $\bar{d} = \bar{d}(x) \in C^\infty(\overline{\Omega})$ is a positive extension from a neighborhood of $\partial\Omega$ to the whole of $\overline{\Omega}$ of the distance function $d(x, \partial\Omega)$.

The spaces $\Xi^s(\Omega)$ with real $s > 0$ are defined by interpolation and then, to $s < 0$ by duality.

Next definition involves uniformly elliptic operators. A differential operator $A(x, \partial) = \sum_{|\alpha|\leqslant 2m} a_\alpha(x)\partial^\alpha$ of order $2m$ and coefficients $a_\alpha \in C^\infty(\overline{\Omega})$ is uniformly elliptic in Ω if a certain positive constant $c > 0$ exists such that

$$\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geqslant c|\xi|^{2m}$$

for all $\xi \in \mathbb{R}^N$, $x \in \Omega$. The following definition is stated in [18] in the context of the broader class of properly elliptic operators.

Definition 5.10. (See [18, p. 199].) Let $A(x, \partial)$ be uniformly elliptic in Ω with order $2m$. For $0 < s < 2m$ we define

$$D_A^s(\Omega) = \left\{u \in H^s(\Omega)\colon Au \in \Xi^{s-2m}(\Omega)\right\},$$

with norm $\|u\|^2_{D_A^s(\Omega)} = \|u\|^2_{H^s(\Omega)} + \|Au\|^2_{\Xi^{s-2m}(\Omega)}$, where $H^s(\Omega)$ stands for the fractional Sobolev space $W^{s,2}(\Omega)$.

In the following results we assume that both the domain Ω and the potential q are of class C^∞. They are particular cases of Theorems 7.3 and 7.4 of Chapter 2 in [18].

Theorem 5.11. *The trace operator extends to a continuous operator from $D_A^s(\Omega)$ to $H^{s-1/2}(\partial\Omega)$.*

Theorem 5.12. *Assume that none of the Neumann eigenvalues of $-\Delta + q$ in Ω is zero. Then, the operator $u \mapsto ((-\Delta + q)u, \frac{\partial u}{\partial \nu})$ is a topological isomorphism from $D_A^s(\Omega)$ into $\Xi^{s-2}(\Omega) \times H^{s-3/2}(\partial\Omega)$, for any $0 < s < 2$.*

Remark 5.13.

a) Conclusion of Theorem 5.12 still holds true if some eigenvalue of $-\Delta + q$ vanishes. In this case such operator still defines an isomorphism if one suitably reduces both its domain and range (see [18]). This fact can be employed to remove the hypothesis on the Neumann invertibility of $-\Delta + q$ from the statement of Theorem 5.1.

b) For $s \geqslant 1$, $((-\Delta + q)u, \frac{\partial u}{\partial \nu}) = (f, g)$ should be understood in the sense that $u \in H^s(\Omega)$ defines a weak solution to the corresponding non homogeneous Neumann problem: $-\Delta u + qu = f$ in Ω, $\partial u/\partial \nu = g$ on $\partial\Omega$.

Proof of Theorem 5.8. Choose $s = \frac{3}{2}$ in Theorems 5.11 and 5.12 and set $u_\varepsilon = \Phi_{1,\varepsilon}$, $u_0 = \Phi_1(0)$ (recall that $m_0 = \chi_\Gamma(x)$). Since $\lambda_{1,\varepsilon} m_\varepsilon u_\varepsilon$, $\lambda_1(0) m_0 u_0 \in L^2(\partial\Omega)$, there exist $\tilde{u}_\varepsilon$, $\tilde{u}_0$ in $D_A^{3/2}(\Omega)$ such that

$$\left(A\tilde{u}_\varepsilon, \frac{\partial \tilde{u}_\varepsilon}{\partial \nu}\right) = (0, \lambda_{1,\varepsilon} m_\varepsilon u_\varepsilon), \qquad \left(A\tilde{u}_0, \frac{\partial \tilde{u}_0}{\partial \nu}\right) = (0, \lambda_0 m_0 u_0).$$

On the other hand $\tilde{u}_\varepsilon$, $\tilde{u}_0$ are weak solutions in $H^1(\Omega)$ to the corresponding nonhomogeneous Neumann problems. Thus $\tilde{u}_\varepsilon = u_\varepsilon$ and $\tilde{u}_0 = u_0$.

In addition, $u_\varepsilon \to u_0$ in $H^1(\Omega)$ (Theorem 5.6) and so $u_\varepsilon \to u_0$ in $L^2(\partial\Omega)$ (indeed, in a more regular subspace). Thus, certain nonnegative $h \in L^2(\partial\Omega)$ exists so that $|u_\varepsilon| \leqslant h$ a.e. on $\partial\Omega$ ([8]). In view of Lemma 5.5, $|\lambda_{1,\varepsilon} m_\varepsilon u_\varepsilon| \leqslant Ch$ a.e. on $\partial\Omega$ for a certain constant $C > 0$. Dominated convergence then yields that the whole family $\lambda_{1,\varepsilon} m_\varepsilon u_\varepsilon$ converges to $\lambda_1(0) m_0 u_0$ in $L^2(\partial\Omega)$. Therefore, Theorem 5.12 implies

$$u_\varepsilon \to u_0 \quad \text{in } D_A^{3/2}(\Omega),$$

and so, convergence assertion (5.10) in Theorem 5.8 directly follows from Theorem 5.11. $\square$

We can already show the main result of this section.

Proof of Theorem 5.1. Relation (5.5) in Theorem 5.6 says

$$\lambda_1'(0) = -\lambda_1(0) \lim_{\varepsilon \to 0} \int_{\partial\Omega} m_\varepsilon \operatorname{div}\left(\Phi_{1,\varepsilon}^2 V\right) d\sigma.$$

Since $\Phi_{1,\varepsilon} \to \Phi_1(0)$ in $H^1(\partial\Omega)$ a nonnegative function $h_1 \in L^1(\partial\Omega)$ exists such that $|\Phi_{1,\varepsilon}| \leqslant h_1$ and $|\Phi_{1,\varepsilon} \partial_i \Phi_{1,\varepsilon}| \leqslant h_1$ a.e. on $\partial\Omega$ for $1 \leqslant i \leqslant N$. Thus, boundedness of m_ε in $L^\infty(\partial\Omega)$ (Lemma 5.5) and dominated convergence permit us to introduce the limit into the integral to achieve

$$\lambda_1'(0) = -\lambda_1(0) \int_{\partial\Omega} m_0 \operatorname{div}\left(\Phi_{1,\varepsilon}^2 V\right) d\sigma = -\lambda_1(0) \int_\Gamma \operatorname{div}\left(\Phi_1(0)^2 V\right) d\sigma.$$

The last integrand lies in $L^1(\partial\Omega)$. Therefore, thanks to the differentiability of Φ_1 on $\partial\Omega \setminus \partial\Gamma$ (Theorem 2.1-iv)) we have

$$\begin{aligned}
\int_\Gamma \operatorname{div}\left(\Phi_1(0)^2 V\right) d\sigma &= \lim_{\delta \to 0+} \int_{\Gamma_\delta^-} \operatorname{div}\left(\Phi_1(0)^2 V\right) d\sigma \\
&= \lim_{\delta \to 0+} \int_{\partial\Gamma_\delta^-} \Phi_1(0)^2 \langle V, \nu_{\partial\Gamma_\delta^-} \rangle \, d\sigma_{\Gamma_\delta^-} = \int_{\partial\Gamma} \Phi_1(0)^2 \langle V, \nu_{\partial\Gamma} \rangle \, d\sigma_\Gamma,
\end{aligned}$$

with $\Gamma_\delta^- = \{x \in \Gamma:\ \operatorname{dist}_{\partial\Omega}(x, \partial\Gamma) > \delta\}$, and where to pass to the limit with δ the fact that $\Phi_1(0) \in C^\beta(\overline{\Omega})$ has been employed. This finishes the proof. $\square$

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