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A BIFURCATION PROBLEM GOVERNED BY THE BOUNDARY CONDITION II

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Abstract

In this work we consider the problem $\Delta u = a(x)u^p$ in Ω, $\frac{\partial u}{\partial \nu} = \lambda u$ on $\partial\Omega$, where Ω is a smooth bounded domain, ν is the outward unit normal to $\partial\Omega$, λ is regarded as a parameter and $0 < p < 1$. We consider both cases where $a(x) > 0$ in Ω or $a(x)$ is allowed to vanish in a whole subdomain Ω_0 of Ω. Our main results include existence of non-negative non-trivial solutions in the range $0 < \lambda < \sigma_1$, where σ_1 is characterized by means of an eigenvalue problem, uniqueness and bifurcation from infinity of such solutions for small λ, and the appearance of dead cores for large enough λ.

1. *Introduction*

The aim of the present work is to complete the study initiated in [**11**] of the non-negative solutions to the following boundary-value problem:

$$\begin{cases} \Delta u = a(x)u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a $C^{2,\gamma}$-domain of $\mathbb{R}^N$, $N \geqslant 2$, $a \in C^\gamma(\overline{\Omega})$ is a non-negative weight function, $p > 0$ and λ is a real parameter. While in [**11**] the case $p > 1$ was treated, we are focusing our attention here in the complementary range $0 < p < 1$. This range is in principle more complex, since standard techniques employed for $p > 1$ are useless here, namely: sub and supersolutions, global minimization, the strong maximum principle and linearization near $u = 0$. For reasons which will become clear later on, this regime is sometimes termed as 'degenerate'.

On the other hand, in reaction diffusion theory, the non-linear term u^p represents the degradation rate of a certain reference reactant whose density is given by u. The exponent p is known as the *order* of the reaction. In this context, the case $0 < p < 1$ is interesting in its own right (see [**3**]).

The main novelty in problem (1.1) is that the parameter appears explicitly in the boundary condition. With respect to this parameter, we will perform a complete analysis of the bifurcation diagram of non-negative solutions to (1.1), which will be shown to be entirely different from the case $p > 1$ (see Figures 1, 2 and 4).

In the course of the exposition, we consider, in the first place, the case of a positive weight $a(x) > 0$ in Ω, and then we will deal with the situation in which $a(x)$ vanishes in a smooth non-empty subdomain Ω_0 of Ω (see [**18**, **6**, **8**, **17**, **10**, **7**] for problems with Dirichlet or Robin boundary conditions, none of them depending on parameters). It should be remarked that dropping the connectedness of Ω_0 does not lead to genuinely new features regarding the material analyzed here and we therefore omit its discussion. Also for the sake of brevity, we restrict our analysis to treat the cases where either Ω_0 is strongly contained in Ω or where Ω_0 'touches'

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$\partial\Omega$ in a non-trivial way (see hypothesis (H)). On the other hand, we point out that due to the smoothness of the main domains Ω, Ω_0 and Ω^+ involved in this work (see details below), their boundaries can only exhibit a finite number of connected components.

An important feature to stress with regard to problem (1.1) is that the strong maximum principle is not applicable and thus non-negative non-trivial solutions need not be positive in Ω. Indeed, it will be seen that solutions u develop a dead core, that is, the set

$$\mathcal{O} = \{x \in \Omega : \, u(x) = 0\}$$

has non-empty interior for large enough λ (see for instance [**3**] or [**9**] for 'dead core' phenomenology), and this entails in some domains multiplicity of solutions. Nevertheless, we can still ensure that solutions are unique for small enough λ and moreover that a bifurcation from infinity at $\lambda = 0$ takes place. Additionally, particular properties of problem (1.1) in the ball are also stated (see below).

We come now to give precise statements of our results. First of all we consider the case of positive weights.

THEOREM 1. *Assume that Ω is a $C^{2,\gamma}$ bounded domain of $\mathbb{R}^N$, $a \in C^\gamma(\overline{\Omega})$, $a(x) > 0$ in Ω and $0 < p < 1$. Then problem* (1.1) *possesses the following features.* (*See Figure* 1.)

(i) *If $\lambda \leqslant 0$, then problem* (1.1) *does not have non-negative non-trivial solutions. For $\lambda > 0$, there is always a non-negative non-trivial solution $u \in C^{2,\gamma_1}(\overline{\Omega})$, with $\gamma_1 = \min\{\gamma, p\}$.*

(ii) *There exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, problem* (1.1) *has a* unique *classical non-negative solution u_λ. In addition, u_λ is positive in that range, $u_\lambda \in C^{2,\gamma}(\overline{\Omega})$ while the mapping $\lambda \to u_\lambda$, regarded as attaining its values in $C^{2,\gamma}(\overline{\Omega})$, is real analytic and decreasing. Moreover,*

$$\lim_{\lambda \to 0+} \lambda^{1/(1-p)} u_\lambda = \left(\frac{1}{|\partial\Omega|} \int_\Omega a(x) \right)^{1/(1-p)}$$

in $C^{2,\gamma}(\overline{\Omega})$. In particular, $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}$ as $\lambda \to 0+$.

(iii) *There exist positive constants C and λ_1 such that for all non-negative non-trivial solutions $u \in C^{2,\gamma_1}(\overline{\Omega})$ to* (1.1) *with $\lambda \geqslant \lambda_1$ we have*

$$u \leqslant C\lambda^{-2/(1-p)}. \tag{1.2}$$

(iv) *There exists $\lambda_2 > 0$ such that all non-negative non-trivial solutions u_λ to* (1.1) *for $\lambda \geqslant \lambda_2$ develop a dead core $\mathcal{O}_\lambda := \{u_\lambda = 0\}$, with $\mathcal{O}_\lambda \to \Omega$ uniformly as $\lambda \to +\infty$ in the sense that for λ large $\{d(x) \geqslant d_\lambda\} \subset \mathcal{O}_\lambda$ with $d_\lambda \to 0$ as $\lambda \to \infty$; here $d(x) = \operatorname{dist}(x, \partial\Omega)$. Moreover, d_λ can be chosen as*

$$d_\lambda = \frac{K}{\lambda},$$

for a certain $K > 0$ provided $a > 0$ on $\partial\Omega$.

Part (iv) of Theorem 1 implies that non-negative solutions are non-trivial only near the boundary of Ω for large λ. By means of a mixed version of problem (1.1) (see (5.1) and Theorem 11) it is possible to show that several solutions can be constructed if we assume that $\partial\Omega$ consists in more than one connected component. We remark that this phenomenon is not present in the equation $\Delta u = \lambda u^p$ if the boundary condition is $u = 1$ on $\partial\Omega$, the problem treated in, for instance, [**3**].

THEOREM 2. *Assume that Ω is a bounded $C^{2,\gamma}$-domain such that $\partial\Omega$ has k connected components. Then problem* (1.1) *has at least $2^k - 1$ non-negative non-trivial solutions for large enough λ.*

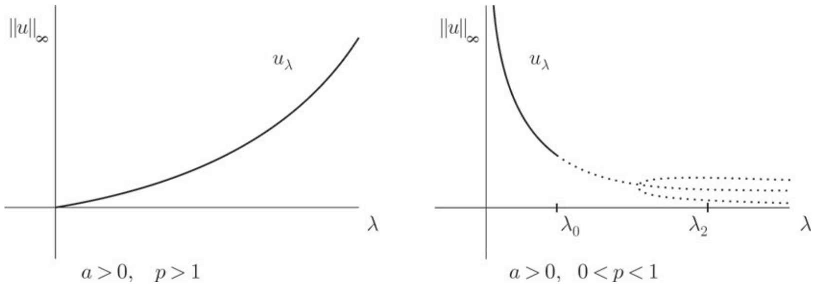

FIGURE 1. *Comparison between the bifurcation diagrams of cases $p > 1$ and $0 < p < 1$ for $a > 0$ in Ω. Continuous pieces in the diagrams stand for uniqueness, the corresponding branches being smooth curves. Multiplicity is vaguely depicted by more than one branch in dotted lines.*

We now turn to consider problem (1.1) in a specific domain: the unit ball B in $\mathbb{R}^N$. In this case one should expect that a radial solution exists provided a is also radial. We show that this is indeed the case, and moreover the radial solution is unique for large λ. It could be thought indeed that there are no non-radial solutions, since in contrast with Theorem 2, ∂B is connected and that result can not be applied. Surprisingly, we prove that this is not the case, by constructing a second solution for large λ which is not radial. For simplicity, we only consider the case $a(x) = 1$.

THEOREM 3. *Assume that $\Omega = B$, $a(x) = 1$ and $0 < p < 1$. Then the following hold.*

(i) *For every $\lambda > 0$, there exists at least one radially symmetric non-negative non-trivial solution to* (1.1).

(ii) *There exist $\lambda_0, \lambda_3 > 0$ such that problem* (1.1) *has a* unique *radially symmetric non-negative non-trivial solution u_λ for both $0 < \lambda < \lambda_0$ and $\lambda > \lambda_3$.*

(iii) *For $\lambda > \lambda_3$ the radial solution u_λ has a dead core $\mathcal{O}_\lambda = \{x \in B : |x| \leqslant r(\lambda)\}$, where*

$$r(\lambda) \sim 1 - \frac{\alpha}{\lambda} \quad \text{as } \lambda \to +\infty$$

and $\alpha = 2/(1-p)$. Moreover, $u_\lambda(1) \sim A\alpha^\alpha \lambda^{-\alpha}$, where $A = [\alpha(\alpha-1)]^{-1/(1-p)}$.

(iv) *There exists $\lambda_4 > 0$ such that problem* (1.1) *has at least one non-negative non-trivial solution v_λ for $\lambda > \lambda_4$ that is not radial.*

Finally, we treat some features of problem (1.1) when the weight function $a(x)$ is allowed to vanish in some non-empty subdomain Ω_0 of Ω. Observe that the connectedness of Ω together with $a \not\equiv 0$ entail $\partial\Omega_0 \cap \Omega \neq \emptyset$. We restrict our discussion here to the case where $\Omega_0 \subset \Omega$ is a $C^{2,\gamma}$ smooth domain such that, defining $\Gamma_1 = \partial\Omega_0 \cap \partial\Omega$ and $\Gamma_2 = \partial\Omega_0 \cap \Omega$, the following condition holds (cf. [**11**]):

$$\overline{\Gamma}_2 \subset \Omega. \tag{H}$$

Condition (H) implies that the part of $\partial\Omega_0$ meeting Ω necessarily consists of a closed manifold ($\overline{\Gamma}_2 = \Gamma_2$) entirely contained in Ω, thus lying at a positive distance from the remaining part Γ_1 of $\partial\Omega_0$, located on $\partial\Omega$. This is a technical hypothesis which provides a convenient smoothness of the eigenfunctions of the auxiliary problems (1.3) and (1.4) (see the discussion after Remark 3 and Remark 4(c)). On the other hand, the separation between Γ_1 and Γ_2 is crucial in the dead core analysis carried out in Theorem 11.

It turns out that the first eigenvalue σ_1 of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_0, \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{1.3}$$

will be determinant in the existence issue of solutions. The main features concerning σ_1 were studied in [**11**] (cf. Theorem 6). Some of them are recalled in Remarks 4 and Theorem 8 of the present work. In such statements, and for our purposes here, problem (1.3) is further analyzed in the more ambitious case where (H) fails and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ (that is, the 'genuine' mixed problem).

To fix the notation, in what follows we set $\sigma_1 = \infty$ if $\Gamma_1 = \emptyset$, that is, $\Omega_0 \subset\subset \Omega$. After these preliminaries, we can state the following result.

THEOREM 4. *Assume that Ω is a $C^{2,\gamma}$ bounded domain of $\mathbb{R}^N$, $a \in C^\gamma(\overline{\Omega})$ and $0 < p < 1$. Then problem* (1.1) *possesses the following features.*

(i) *Problem* (1.1) *does not have non-negative non-trivial solutions when $\lambda \leqslant 0$. For $0 < \lambda < \sigma_1$, there always exists a non-negative non-trivial solution $u \in C^{2,\gamma_1}(\overline{\Omega})$, where $\gamma_1 = \min\{p, \gamma\}$.*

(ii) *There exists λ_0, with $0 < \lambda_0 < \sigma_1$, such that for $0 < \lambda < \lambda_0$, problem* (1.1) *has a* unique *classical non-negative solution u_λ. The mapping $\lambda \to u_\lambda$ exhibits the same properties as in* Theorem 1(ii). *In particular,*

$$\lim_{\lambda \to 0+} \lambda^{1/(1-p)} u_\lambda = \left(\frac{1}{|\partial\Omega|} \int_\Omega a(x) \right)^{1/(1-p)}$$

in $C^{2,\gamma}(\overline{\Omega})$ and so $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}$ as $\lambda \to 0+$.

(iii) *If $\sigma_1 = +\infty$, then there exist positive constants C, K and λ_2 such that for all non-negative non-trivial solutions $u_\lambda \in C^{2,\gamma_1}(\overline{\Omega})$ to* (1.1) *with $\lambda \geqslant \lambda_2$ we have*

$$u_\lambda \leqslant C\lambda^{-2/(1-p)}.$$

In addition u develops a dead core $\mathcal{O}_\lambda$ with $\{\operatorname{dist}(x, \partial\Omega) \geqslant d_\lambda\} \subset \mathcal{O}_\lambda$ where $d_\lambda \to 0+$ as $\lambda \to +\infty$. Moreover, $d_\lambda = K/\lambda$, for a constant $K > 0$, provided $a > 0$ on Γ_1.

An important difference of problem (1.1) in the range $0 < p < 1$ compared to $p > 1$ arises when $\lambda \geqslant \sigma_1$. Specifically, it was proved in [**11**] that no positive solutions of (1.1) with $p > 1$ exist when $\lambda \geqslant \sigma_1$, provided $\sigma_1 < +\infty$ (see Figure 2). We show next that this is indeed the case if $\Omega^+ := \{x \in \Omega : a(x) > 0\} \subset\subset \Omega$, but things are quite different otherwise.

In explaining the discrepancy when $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ the eigenvalue problem (1.3), now observed in Ω^+, again has an important rôle. Let $\{\Omega_i^+\}$ be the set of (finitely many) connected pieces of Ω^+. Observe that $\partial\Omega^+ \cap \Omega = \partial\Omega_0 \cap \Omega = \Gamma_2$ (as defined before) and so $\Gamma_2 = \bigcup_i (\partial\Omega_i^+ \cap \Omega)$. Notice again that, due to the fact that $a \not\equiv 0$ and the connectedness of Ω, each $\Gamma_{2,i} := \partial\Omega_i^+ \cap \Omega$ is non-empty. In addition $\Gamma^+ := \partial\Omega^+ \cap \partial\Omega = \partial\Omega \setminus \Gamma_1 = \bigcup_i (\partial\Omega_i^+ \cap \partial\Omega)$. If $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ then some $\partial\Omega_i^+$ meets $\partial\Omega$. Precisely, for all those components Ω_i^+ define $\sigma = \tilde{\sigma}_{1,i}$ as the first eigenvalue to the problem

$$\begin{cases} \Delta u = 0 & \text{for } x \in \Omega_i^+, \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{for } x \in \partial\Omega_i^+ \cap \partial\Omega, \\ u = 0 & \text{for } x \in \Gamma_{2,i} = \partial\Omega_i^+ \cap \Omega. \end{cases} \tag{1.4}$$

Designate $\tilde{\sigma}_1 = \min \tilde{\sigma}_{1,i}$ if $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ and set $\tilde{\sigma}_1 = +\infty$ otherwise (that is, the case $\partial\Omega = \Gamma_1$).

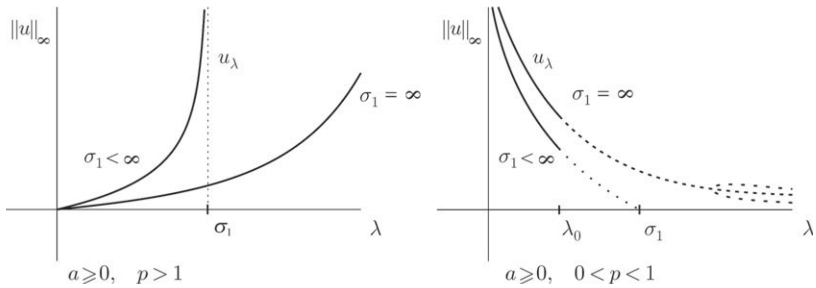

FIGURE 2. *Bifurcation diagrams $p > 1$ versus $0 < p < 1$. In both regimes drawings for cases $\sigma_1 = \infty$ and $\sigma_1 < \infty$ but $\tilde{\sigma}_1 = \infty$ are superposed. The meaning of continuous and dotted or dashed arcs is the same as in Figure 1.*

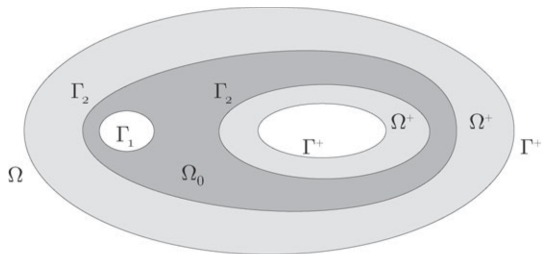

FIGURE 3. *A possible configuration for the domains Ω, Ω_0 and Ω^+. In this example, Ω^+ possesses two connected pieces.*

The new features concerning (1.1) in the case where a vanishes in Ω are described next. (See Figure 3.)

THEOREM 5. *Assume that $\sigma_1 < +\infty$. Then the following hold.*

(i) *If $\tilde{\sigma}_1 = +\infty$, then there are no non-negative non-trivial solutions to* (1.1) *with $\lambda \geqslant \sigma_1$.*

(ii) *If $\tilde{\sigma}_1 < +\infty$ and there exists a non-negative non-trivial solution to* (1.1) *with $\lambda \geqslant \sigma_1$, then $\lambda > \tilde{\sigma}_1$. In particular, if $\tilde{\sigma}_1 \geqslant \sigma_1$, then there are no solutions for $\lambda \in [\sigma_1, \tilde{\sigma}_1]$.*

(iii) *When $\tilde{\sigma}_1 < +\infty$, there exists $\lambda_2 > 0$ such that for $\lambda \geqslant \lambda_2$ problem* (1.1) *has at least one non-negative non-trivial solution, which develops a dead core $\mathcal{O}_\lambda$. In this case $\{\operatorname{dist}(x, \Gamma^+) \geqslant d_\lambda\} \subset \mathcal{O}_\lambda$ with $d_\lambda \to 0+$ as $\lambda \to +\infty$, and $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$. Similarly, the choice $d_\lambda = K/\lambda$, with $K > 0$, is again possible when $a > 0$ on Γ^+.*

REMARKS 1. (a) By means of examples it is possible to show that both options $\sigma_1 \leqslant \tilde{\sigma}_1$ and $\sigma_1 > \tilde{\sigma}_1$ can occur. Notice that such relative positions depend only on the support of the weight a, not on its size. See Remark 8.

(b) Regarding (ii) it is also possible to produce examples where $\tilde{\sigma}_1 < \sigma_1$ and either no solutions $u \geqslant 0$ ($u \neq 0$) exist for $\sigma_1 \leqslant \lambda \leqslant \sigma_1 + \varepsilon$, or a non-negative non-trivial solution exists for all $\lambda \geqslant \sigma_1 - \varepsilon$, with $\varepsilon > 0$ small. See also Remark 8.

(c) As already noticed and in contrast with the case $p > 1$, (1.1) can support non-negative non-trivial solutions for $\lambda > \sigma_1$, provided $\tilde{\sigma}_1 < \sigma_1$. However, such solutions are 'degenerate' in

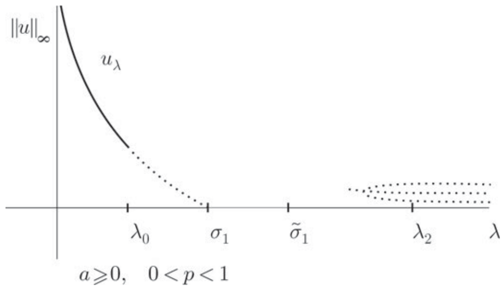

FIGURE 4. *A bifurcation diagram for the case $\sigma_1 < \tilde{\sigma}_1 < +\infty$. Solutions disappear converging to zero at σ_1 and spontaneously appear beyond $\lambda = \tilde{\sigma}_1$.*

the sense that they must vanish in Ω_0 together with all those (if any) connected pieces Ω_i^+ of Ω^+ such that $\Omega_i^+ \subset\subset \Omega$ (cf. § 7).

(d) According to (ii) and (iii) the set of solutions to (1.1) undergoes a discontinuity at $\lambda = \sigma_1$ to, roughly speaking, arise again at some λ after $\tilde{\sigma}_1$ (suppose $\sigma_1 \leqslant \tilde{\sigma}_1$). It should be remarked that the latter arising is necessarily 'spontaneous' in the sense that when solutions appear they are bounded away from zero. In other words, they are not generated by a bifurcation from the trivial solution $u = 0$. See Remark 7 and Figure 4.

Finally, we conclude our analysis by determining the asymptotic behavior of solutions near $\lambda = \sigma_1$ in the critical situation where solutions cease to exist at that value. We assert that $(\lambda, u) = (\sigma_1, 0)$ defines in that case a bifurcation point for solutions to (1.1).

THEOREM 6. *Assume that $\sigma_1 < +\infty$, with either $\sigma_1 \leqslant \tilde{\sigma}_1$ or $\tilde{\sigma}_1 = +\infty$, or even $\tilde{\sigma}_1 < \sigma_1$ but in this case also supposing that* (1.1) *has only $u = 0$ as a solution at $\lambda = \sigma_1$. Then every family of non-negative solutions $\{u_\lambda\}$ with λ close to σ_1 satisfies*

$$u_\lambda \to 0,$$

in $C^{2,\beta}(\overline{\Omega})$, with $0 < \beta < \gamma_1$, as $\lambda \to \sigma_1-$. In addition, there exists $\delta > 0$ such that every non-negative non-trivial solution u_λ to (1.1) *for $\sigma_1 - \delta \leqslant \lambda < \sigma_1$ develops a dead core $\mathcal{O}_\lambda$. More importantly, such dead core must satisfy $\mathcal{O}_\lambda \subset \Omega^+$ together with $\mathcal{O}_\lambda \to \Omega^+$ uniformly as $\lambda \to \sigma_1+$.*

REMARK 2. Compared with the case $\lambda \geqslant \sigma_1$ (Remark 1(c)) it follows from the preceding statement that non-negative solutions to (1.1) with $\lambda < \sigma_1$, $\lambda \sim \sigma_1$, exhibit the opposite behavior in Ω_0. Namely, all those solutions are strictly positive in Ω_0, apart from converging to zero as $\lambda \to \sigma_1-$.

The paper is organized as follows: in § 2, we state some preliminaries which will be used in the paper. Section 3 is devoted to the existence of non-negative solutions and uniqueness for small λ, while in § 4 the asymptotic behavior as $\lambda \to +\infty$ is elucidated. In §§ 5 and 6 we prove Theorems 2 and 3, respectively. A mixed problem, closely related to (1.1) is also studied in § 5 (Theorem 11). Finally, § 7 deals with problem (1.1) when the weight $a(x)$ vanishes in a subdomain Ω_0 of Ω (Theorems 4, 5 and 6).

2. *Preliminaries*

In this section we collect some results which will be needed for the proofs of our theorems. We begin by proving that weak solutions to (1.1) are indeed classical solutions. It should be stressed that this result is not contained in [**1**]. However, as will be seen at once, it can be proved by means of their estimates.

LEMMA 7. *Let $u \in H^1(\Omega)$ be a weak non-negative solution to* (1.1). *Then $u \in C^{2,\gamma_1}(\overline{\Omega})$, with $\gamma_1 = \min\{\gamma, p\}$, and thus defines a classical solution.*

Proof. For $\lambda > 0$ fixed and $\mu > 0$ large enough, the problem

$$\begin{cases} \Delta v - \mu v = f & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \lambda v & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

with $f = 0$ has $u = 0$ as the only weak solution in $H^1(\Omega)$. In fact, any non-trivial solution $u \in H^1(\Omega)$ to (2.1) with $f = 0$ defines a weak eigenfunction corresponding to the eigenvalue $\sigma = \lambda$ of the Steklov-type eigenvalue problem

$$\begin{cases} \Delta v - \mu v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \sigma v & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Techniques in [**11**, § 2] can be used to show that problem (2.2) possesses, for every μ, a *first* eigenvalue $\sigma_1 = \sigma_1(\mu)$ which is the unique eigenvalue associated to a positive eigenfunction in $H^1(\Omega)$ (*a posteriori* in $C^{2,\gamma}(\overline{\Omega})$) and variationally characterized as

$$\sigma_1 = \inf_{v \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \{|\nabla v|^2 + \mu v^2\}}{\int_{\partial\Omega} v^2}. \tag{2.3}$$

Moreover, σ_1 increases with μ. We claim that $\sigma_1 \to \infty$ as $\mu \to \infty$. This implies the desired assertion if μ is so large as to have $\lambda < \sigma_1(\mu)$. To show the claim suppose that $\sigma_1(n) = O(1)$ as $n \to \infty$. If $\phi_n \in H^1(\Omega)$ stands for the positive eigenfunction corresponding to $\sigma_1(n)$ normalized so that $\int_{\partial\Omega} \phi_n^2 = 1$, then ϕ_n remains bounded in $H^1(\Omega)$ under the norm

$$|u|_{H^1}^2 = \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2.$$

Modulo a subsequence, $\phi_n \to \phi$ weakly in H^1 and by compactness $\phi_n \to \phi$ in both $L^2(\Omega)$ and $L^2(\partial\Omega)$. In particular $\int_{\partial\Omega} \phi^2 = 1$. However, the boundedness of $\sigma_1(n)$ leads to $\int_\Omega \phi^2 = 0$ which is not possible. Thus the claim is proved.

Since (2.1) can admit at most one classical solution, the results in [**14**] (Theorem 6.31 and the subsequent remark) ensure that such a problem does indeed have a unique solution $v \in C^{2,\gamma}(\overline{\Omega})$ for every $f \in C^\gamma(\overline{\Omega})$.

Now let $u \in H^1(\Omega)$ be a non-negative solution to (1.1). By the Sobolev embedding, we have $u, u^p \in L^r(\Omega)$ for every $1 \leqslant r \leqslant 2^* = 2N/(N-2)$ (this is only true if $N \geqslant 3$; for $N = 2$ the situation is even better). Take a sequence of C^γ-functions f_n converging to $a(x)u^p - \mu u$ in $L^r(\Omega)$, and let v_n be the unique solution to (2.1) with $f = f_n$. We now use the estimates of Agmon, Douglis and Nirenberg [**1**]. There exists a constant $C > 0$ such that

$$|v_n|_{W^{2,r}} \leqslant C |f_n|_{L^r}.$$

It follows that $v_n \to v$ in $W^{2,r}(\Omega)$, where v is the (unique) solution to

$$\begin{cases} \Delta v - \mu v = a(x)u^p - \mu u & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \lambda v & \text{on } \partial\Omega. \end{cases}$$

Thus $v = u$, and so $u \in W^{2,r}(\Omega)$. Repeating this argument a finite number of times, we arrive at $u \in W^{2,q}(\Omega)$ for some $q > N$, and hence $u \in C^{1,\eta}(\overline{\Omega})$ for a certain $0 < \eta < 1$.

Next we choose a sequence $f_n \in C^{\gamma_1}(\overline{\Omega})$ so that $f_n \to a(x)u^p - \mu u$ in $C^{\gamma_1}(\overline{\Omega})$, and denote by v_n the $C^{2,\gamma_1}(\overline{\Omega})$ solutions to (2.1) corresponding to $f = f_n$. We now use the Schauder estimates provided by Theorem 6.30 in [**14**] to obtain

$$|v_n|_{C^{2,\gamma_1}} \leqslant C(|f_n|_{C^{\gamma_1}} + |v_n|_{C^{\gamma_1}}).$$

From this it follows similarly that $v_n \to u$ in $C^{2,\gamma_1}(\overline{\Omega})$, and thus $u \in C^{2,\gamma_1}(\overline{\Omega})$. □

REMARK 3. Observe that as a consequence of the last part of the proof of Lemma 7 *positive* solutions u to (1.1) are slightly more regular and lie directly in $C^{2,\gamma}(\overline{\Omega})$, while this is not necessarily the case if such solutions vanish somewhere in Ω.

We now introduce an eigenvalue problem under mixed boundary conditions of Dirichlet and Steklov type, located on zones Γ and Γ' of the boundary, which will play an important rôle in some of our forthcoming proofs. Theorem 8 in [**11**] analyzed this problem in full detail when the zones carrying different boundary conditions are *different* (and hence separated away) components of the boundary, providing class $C^{2,\gamma}$ eigenfunctions. This is precisely the version used most frequently in this work. Our next result deals with the more adverse situation in which Γ and Γ' meet on an $(N-2)$-dimensional closed manifold γ. Such a scenario will be required for showing the estimate (1.2) in § 4. Notice that in this case results in [**11**, § 2] still provide H^1-eigenfunctions (see Remark 8 there). However, their optimal degree of regularity is an issue belonging to the subtle realm of smoothness of weak solutions to mixed problems (see for instance [**15**, **16**]). Accordingly, we only provide next the amount of weak smoothness strictly required for our purposes here. Its proof, a straightforward consequence of [**11**, § 2] and [**4**], is only sketched.

THEOREM 8. *Let $D \subset \mathbb{R}^N$ be a bounded domain of class C^3 such that $\partial D = \Gamma \cup \Gamma' \cup \gamma$, where Γ, Γ' and γ are pair-wise disjoint, Γ and Γ' are relatively open in ∂D, and γ is a closed $(N-2)$-dimensional manifold while $\Gamma \cup \gamma$ and $\Gamma' \cup \gamma$ define $(N-1)$-dimensional manifolds of class C^3 with common boundary γ. Then the eigenvalue problem*

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \Gamma, \\ u = 0 & \text{on } \Gamma', \end{cases} \tag{2.4}$$

admits a principal eigenvalue σ_1, that is, an eigenvalue with a one-signed eigenfunction, which is characterized by

$$\sigma_1 = \inf_{u \in H^1_{\Gamma'}(D)} \frac{\int_D |\nabla u|^2}{\int_\Gamma u^2}, \tag{2.5}$$

where $H^1_{\Gamma'}(D)$ is the subset of functions of $H^1(D)$ which vanish on Γ'. Moreover, σ_1 is the unique principal eigenvalue; it is simple and the smallest of all the eigenvalues. Moreover, every associated eigenfunction ϕ satisfies $\phi \in H^1(D) \cap W^{2,q}(D)$ for all $q < 4/3$.

Sketch of the proof. Observe that $H^1_{\Gamma'}(D)$ is well defined even if Γ and Γ' intersect. Consider

$$\mathcal{M} = \left\{ u \in H^1_{\Gamma'}(D) : \int_\Gamma u^2 = 1 \right\}$$

and set

$$J(u) = \int_D |\nabla u|^2.$$

Regarding $H^1_{\Gamma'}(D)$ endowed with the equivalent norm $|u|_1 = \left(\int_D |\nabla u|^2 \right)^{1/2}$ we see that J is coercive, that is, $J(u_n) \to +\infty$ whenever $|u_n|_1 \to +\infty$. Since J is also weakly lower semicontinuous and $\mathcal{M}$ is weakly closed, the basic result in the calculus of variations (see [**20**]) provides a global minimizer $\phi \in H^1_{\Gamma'}(D)$, which is an eigenfunction of (2.4) associated to σ_1.

It can be assumed that $\phi^+ = \max\{\phi, 0\} \not\equiv 0$, and hence it is also a minimizer. Thus ϕ^+ is harmonic in Ω and by the Harnack inequality we have $\phi^+ > 0$, and hence $\phi > 0$. By an orthogonality argument, it follows that σ_1 is simple, and it is the only eigenvalue associated to a positive eigenfunction. Finally, the extra regularity of the eigenfunction is a consequence of [**4**, Theorem B].

Remarks 4. (a) For $\lambda > 0$ designate by D_λ the scaled copy of D, that is,

$$D_\lambda = \{\lambda x : x \in D\}.$$

If $\sigma_1(D_\lambda)$ stands for the principal eigenvalue of (2.4) in D_λ then one finds directly that $\sigma_1(D_\lambda) = \lambda^{-1}\sigma_1(D)$.

(b) A suitable use of the L^p-estimates in [**1**] proves that every eigenfunction ϕ associated to σ_1 satisfies $\phi \in C^{2,\eta}(D \cup \overline{T})$, with $0 < \eta < 1$ arbitrary, where T is any relatively open part of ∂D strongly contained in either Γ or Γ'. Thus, as a consequence of Hopf's maximum principle, every positive eigenfunction satisfies $\frac{\partial \phi}{\partial \nu} < 0$ on Γ' where ν is the outward unit normal on Γ'. On the other hand, although not in our framework here, smoothness of such eigenfunctions ϕ is enlarged provided Γ and Γ' meet in a convenient non-smooth way on γ, for instance, under a suitable, not too large, angle (cf. [**15**, **16**] and their references). In any case, it should be recalled that eigenfunctions achieve the full $C^{2,\gamma}$-regularity assuming that $\partial\Omega$ is *only* $C^{2,\gamma}$ when Γ and Γ' are *disjoint* closed manifolds (see [**11**]).

(c) The hypothesis that D is of class C^3 is required in [**4**] for the $W^{2,q}$-regularity of the eigenfunctions. On the other hand, the exponent 4/3 for the integrability of the second derivatives is somehow optimal, as an example in [**19**] shows.

Finally, we introduce some results from [**12**] concerning a singular initial value problem, which will be needed in § 6. It should be remarked that such results were established there for the harder framework of the p-Laplacian operator. As a very special case of them we consider the Cauchy problem

$$\begin{cases} ((r+d)^{N-1}u')' = (r+d)^{N-1}u^p & \text{for } r \in (0,\infty), \\ u(0) = 0, \quad u'(0) = 0, \end{cases} \tag{2.6}$$

with $d \geqslant 0$, which arises after some normalization when one considers the radial version of (1.1) with $a(x) = 1$. As a consequence of Theorems 2.3, 2.5 and 2.6 (see also Corollary 2.4) in [**12**] and a globalizing argument, we can state the following theorem.

Theorem 9. *For each $d \geqslant 0$, problem* (2.6) *has a unique* non-trivial *solution $u(r,d)$ defined in* $[0,+\infty)$, *in the sense that $u(r,d) > 0$ for $r > 0$. Moreover, $u(\cdot,d) \to u_0(\cdot)$ as $d \to \infty$ in* $C^1_{\text{loc}}[0,+\infty)$, *where $u_0(\cdot)$ is the non-trivial solution to* (2.6) *corresponding to $N = 1$, which is* *explicitly given by*

$$u_0(r) = Ar^\alpha,$$

where $\alpha = 2/(1-p)$ *and* $A = (\alpha(\alpha-1))^{-1/(1-p)}$. *In addition,* $u(r,d)$ *is differentiable with respect to* d *and satisfies*

$$\frac{\partial u}{\partial d}(\cdot, d) \to 0, \quad \text{as } d \to \infty, \tag{2.7}$$

in $C^1_{\mathrm{loc}}[0,+\infty)$.

Remark 5. Notice that problem (2.6) always has the trivial solution $u \equiv 0$. It follows that this is the only non-negative solution when $p \geqslant 1$. However, when $0 < p < 1$ there are a *unique* positive solution and infinitely many non-negative (non-trivial) solutions, all of them expressible in terms of the positive one (cf. [**12**, Corollary 2.4]).

3. *Existence of non-negative solutions. Uniqueness for small* λ

In this section we consider the issue of the existence of non-negative non-trivial solutions to (1.1) when $a(x) > 0$ in Ω. It will be shown that for every $\lambda > 0$ there always exists at least one solution (while for $\lambda \leqslant 0$ they cease to exist). In addition, we will show that the solution is unique provided λ is sufficiently small. In fact, a bifurcation from infinity at $\lambda = 0$ takes place.

Proof of Theorem 1(i). According to Lemma 7, all weak solutions in $H^1(\Omega)$ belong to $C^{2,\gamma_1}(\overline{\Omega})$. Assume first that $\lambda \leqslant 0$. Integrating the equation (1.1) in Ω, we have

$$\int_\Omega a(x)u^p = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \lambda \int_{\partial\Omega} u \leqslant 0,$$

from which $u \equiv 0$ follows.

Now assume $\lambda > 0$. We use in $H^1(\Omega)$ the norm $|u|^2_{H^1} = \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2$ which is equivalent to the usual one. Define

$$M := \left\{ u \in H^1(\Omega) : \int_\Omega a(x)|u|^{p+1} = 1 \right\}$$

and for $u \in M$ the functional

$$J(u) = \int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2.$$

We claim that J is coercive on M. Indeed, assume on the contrary that there exists a sequence $\{u_n\} \subset M$ such that $|u_n|_{H^1} \to +\infty$ and $J(u_n) \leqslant C$, that is,

$$\int_\Omega |\nabla u_n|^2 \leqslant \lambda \int_{\partial\Omega} u_n^2 + C. \tag{3.1}$$

Denoting $t_n = |u_n|_{L^2(\partial\Omega)}$, we see from (3.1) that $t_n \to +\infty$. Letting $v_n = u_n/t_n$, we obtain again, from (3.1), a bound for the H^1-norms of v_n. Passing to a subsequence, we obtain $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and in $L^2(\partial\Omega)$. In particular, $|v|_{L^2(\partial\Omega)} = 1$. On the other hand, since $p < 1$ and $u_n \in M$, we have $\int_\Omega a(x)|v|^{p+1} = 0$, and so $v = 0$, a contradiction. Thus J is coercive.

Since M is weakly closed, and J is (sequentially) weakly lower semicontinuous, it follows from standard results that J achieves its minimum in M (see [**20**]). Also, since $J(|u|) = J(u)$ and $|u| \in M$ whenever $u \in M$, we may assume that the minimum is achieved at a non-negative (non-trivial) function u. By the Lagrange multipliers rule, there exists $\mu \in \mathbb{R}$ such that

$$\int_\Omega \nabla u \nabla \varphi - \lambda \int_{\partial\Omega} u\varphi = \mu \int_\Omega a(x) u^p \varphi$$

for every $\varphi \in H^1(\Omega)$. Taking, in particular, $\varphi = u$, we obtain $J(u) = \mu < 0$, since there is a constant function which belongs to M for which J is negative. Setting $v = |\mu|^{1-p} u$, we obtain a non-negative non-trivial weak solution to (1.1), which is, as remarked earlier, a classical solution in $C^{2,\gamma_1}(\overline{\Omega})$. □

Remark 6. Non-negative solutions can also be obtained by applying the Mountain Pass theorem to the functional

$$\tilde{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega} (u_+)^2 + \frac{1}{p+1}\int_\Omega a(x)|u|^{p+1},$$

in $H^1(\Omega)$, where $u_+ = \max\{u, 0\}$ denotes the positive part of u.

We now proceed to show Theorem 1(ii). The proof makes use of local bifurcation theory. As a first step in this approach, we begin by characterizing the behavior of all non-negative solutions for λ approaching zero.

Lemma 10. *Assume $\lambda_n \to 0$, and let u_n be a corresponding sequence of non-negative non-trivial solutions to* (1.1). *Then*

$$\begin{cases} \lambda_n = \mu_n t_n, \\ u_n = \dfrac{1}{t_n^\theta}(1 + t_n w_n), \end{cases}$$

where $\theta = 1/(1-p)$,

$$t_n = \left(\frac{1}{|\partial\Omega|} \int_{\partial\Omega} u_n \right)^{p-1} \to 0 \quad \text{and} \quad w_n \in Y := \left\{ u \in C^{2,\gamma}(\overline{\Omega}) : \int_{\partial\Omega} u = 0 \right\}.$$

Moreover,

$$\mu_n \to \mu_0 = \frac{1}{|\partial\Omega|} \int_\Omega a(x), \tag{3.2}$$

and $w_n \to w_0$ in $C^{2,\beta}(\overline{\Omega})$ for all $0 < \beta < \gamma$, where w_0 is the unique solution in Y of the linear equation

$$\begin{cases} \Delta w = a(x) & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = \mu_0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Proof. First of all, notice that, according to the compatibility condition, (3.3) has a unique solution in Y given by the value of μ_0 in (3.2).

We begin by showing that $|u_n|_\infty$ tends to infinity. Assume on the contrary that for a subsequence (still labelled by u_n) we have $|u_n|_\infty \leqslant C$. Observing that problem (2.1) corresponding to $\mu = 0$ has a unique solution for λ small and by employing the Agmon–Douglis–Nirenberg estimates (see [**1**]), we have for every $q > 1$,

$$|u_n|_{W^{2,q}} \leqslant C_1(|u_n|_{L^q} + |u_n^p|_{L^q}) \leqslant C_2, \tag{3.4}$$

where $C_1 > 0$ is a certain constant depending on q. Selecting $q > N$, we have by the Sobolev imbedding (passing to a subsequence) that $u_n \to u$ in $C^{1,\eta}(\overline{\Omega})$ for some $0 < \eta < 1$. It follows that u is a non-negative weak (and hence classical by Lemma 7) solution to (1.1) with $\lambda = 0$. Thus, $u \equiv 0$ (in particular, $u_n \to 0$ in $H^1(\Omega)$). We claim that this is impossible. Indeed, let

$v_n = u_n/|u_n|_{H^1}$. There exists a subsequence (labelled once more by v_n) such that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$, in $L^2(\partial\Omega)$ and almost everywhere in Ω. It follows from (1.1) that

$$\int_\Omega |\nabla v_n|^2 = \lambda_n \int_{\partial\Omega} v_n^2 - |u_n|_{H^1}^{p-1} \int_\Omega a(x) v_n^{p+1} \leqslant \lambda_n \int_{\partial\Omega} v_n^2 \to 0,$$

so by lower semicontinuity we deduce that v is a constant. As a consequence, it follows that $v_n \to v$ *strongly* in $H^1(\Omega)$. On the other hand, again by (1.1), we have

$$|u_n|_{H^1}^{p-1} \int_\Omega a(x) v_n^{p+1} = \lambda_n \int_{\partial\Omega} v_n^2 - \int_\Omega |\nabla v_n|^2 \to 0,$$

and since $0 < p < 1$, we deduce that

$$\int_\Omega a(x) v_n^{p+1} \to 0. \tag{3.5}$$

Now since $v_n \to v$ in $L^2(\Omega)$, we have $v_n \to v$ in $L^{p+1}(\Omega)$, and thus $v = 0$ in Ω, contradicting the fact that $|v_n|_{H^1} = 1$. Thus $|u_n|_\infty \to +\infty$.

Setting again $v_n = u_n/|u_n|_\infty$, and recalling that v_n satisfies

$$\begin{cases} \Delta v_n = a(x) v_n^p |u_n|_\infty^{p-1} & \text{in } \Omega, \\ \dfrac{\partial v_n}{\partial \nu} = \lambda_n v_n & \text{on } \partial\Omega, \end{cases}$$

we obtain, arguing as before, an estimate like (3.4), thus getting a bound in $C^{1,\eta}(\overline{\Omega})$ for the solutions v_n for some $0 < \eta < 1$. Applying [**14**, Theorem 6.30] gives $C^{2,\gamma_1}(\overline{\Omega})$-bounds, and thus $v_n \to v$ in $C^{2,\beta}(\overline{\Omega})$, with $0 < \beta < \gamma_1$, where v is a unique classical solution to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $|v|_\infty = 1$. Hence $v \equiv 1$. Since v_n becomes positive for large n, we find, furthermore, that $v_n \in C^{2,\gamma}(\overline{\Omega})$ (Remark 3) and $v_n \to 1$ in $C^{2,\beta}(\overline{\Omega})$ for all $0 < \beta < \gamma$.

We now split $u_n = c_n + c_n z_n$, where $c_n = (1/|\partial\Omega|) \int_{\partial\Omega} u_n$ and $z_n \in Y$. It follows that

$$\frac{c_n}{|u_n|_\infty} \to 1 \quad \text{and} \quad z_n \to 0 \quad \text{in } C^{2,\beta}(\overline{\Omega}).$$

Next, notice that z_n solves

$$\begin{cases} \Delta z_n = a(x) c_n^{p-1} (1 + z_n)^p & \text{in } \Omega, \\ \dfrac{\partial z_n}{\partial \nu} = \lambda_n (1 + z_n) & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

By integrating in (3.6) we obtain

$$\int_\Omega a(x)(1 + z_n)^p = \lambda_n c_n^{1-p} |\partial\Omega|.$$

Setting $t_n = c_n^{p-1}$ we see that this means that $\lambda_n = t_n \mu_n$ where $\mu_n \to \mu_0$, with μ_0 given by (3.2). Next, using the fact that $\lambda_n = O(t_n)$ and from Schauder's estimates, we conclude that $z_n = O(t_n)$ in $C^{2,\gamma}(\overline{\Omega})$. Writing $z_n = t_n w_n$ with w_n bounded in $C^{2,\gamma}(\overline{\Omega})$ we conclude that $w_n \to w_0$ in $C^{2,\beta}(\overline{\Omega})$, w_0 being the unique classical solution in Y to (3.3). This proves the lemma. □

Proof of Theorem 1(ii). According to Lemma 10, all non-negative non-trivial solutions to (1.1) for small λ are of the form

$$\begin{cases} \lambda = t\mu, \\ u = \dfrac{1}{t^\theta}(1+tw), \end{cases}$$

where $t>0$ is small and $(\mu, w) \in \mathbb{R} \times Y$ is close to (μ_0, w_0), with $\mu_0 \in \mathbb{R}$ and $w_0 \in Y$ given by (3.2) and (3.3) respectively. Thus, for small $\lambda > 0$, solving (1.1) is equivalent to solving

$$\begin{cases} \Delta w = a(x)(1+tw)^p & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = \mu(1+tw) & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

together with the compatibility condition

$$\int_\Omega a(x)(1+tw)^p = \mu \int_{\partial\Omega} (1+tw), \tag{3.8}$$

for $|t|$, $|\mu - \mu_0|$ and $|w - w_0|_Y$ small, where $Y = \left\{u \in C^{2,\gamma}(\overline{\Omega}) : \int_{\partial\Omega} u = 0\right\}$ is endowed with its natural norm.

We accordingly show such uniqueness by means of the Implicit Function theorem as follows. Setting $X = C^\gamma(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$, define $l \in X^*$ as

$$l(f,g) = \int_\Omega f - \int_{\partial\Omega} g, \quad \text{for } (f,g) \in X.$$

According to Schauder's theory, for each $(f,g) \in \ker(l)$ the problem

$$\begin{cases} \Delta u = f & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{for } x \in \partial\Omega, \end{cases}$$

has a unique solution $u \in Y$, denoted $u = K(f,g)$. Hence, $K : \ker(l) \to Y$ defines a linear topological isomorphism. Observe, in addition, that $(f,g) \mapsto (f, g + (1/|\partial\Omega|)l(f,g))$ defines a projection from X onto $\ker(l)$. Thus K is always defined at $(f, g + (1/|\partial\Omega|)l(f,g))$ for every $(f,g) \in X$.

Now introduce the Nemytskii operators $F(t,w) = a(1+tw)^p$ and $G(t,w) = 1+tw$. For small and positive ε and δ, the maps

$$F : (-\varepsilon, \varepsilon) \times B_Y(w_0, \delta) \longrightarrow C^\gamma(\overline{\Omega}) \quad \text{and} \quad G : (-\varepsilon, \varepsilon) \times B_Y(w_0, \delta) \longrightarrow C^{1,\gamma}(\overline{\Omega})$$

define real analytic mappings; here $B_Y(w_0, \delta)$ stands for the open ball in Y with radius δ and center w_0. Thus,

$$\begin{aligned} \mathcal{F} : (-\varepsilon, \varepsilon) \times (\mu_0 - \varepsilon, \mu_0 + \varepsilon) \times B_Y(w_0, \delta) &\longrightarrow \mathbb{R} \times Y, \\ (t, \mu, w) &\longmapsto (\mathcal{F}_1(t,\mu,w), \mathcal{F}_2(t,\mu,w)), \end{aligned}$$

where

$$\mathcal{F}_1(t,\mu,w) = l(F(t,w), \mu G(t,w)),$$

and

$$\mathcal{F}_2(t,\mu,w) = w - K\left(F(t,w), \mu G(t,w) + \frac{1}{|\partial\Omega|}\mathcal{F}_1(t,\mu,w)\right)$$

is also a real analytic mapping.

On the other hand, solving (3.7) and (3.8) for (t,μ,w) close to $(0,\mu_0,w_0)$ in $\mathbb{R}^2\times Y$ is equivalent to solving

$$\mathcal{F}(t,\mu,w)=0, \tag{3.9}$$

with $(t,\mu,w)\in(-\varepsilon,\varepsilon)\times(\mu_0-\varepsilon,\mu_0+\varepsilon)\times B(w_0,\delta)$ with ε and δ small enough. However, $\mathcal{F}(0,\mu_0,w_0)=0$, while

$$\mathcal{F}_1(0,\mu,w)=|\partial\Omega|(\mu_0-\mu),$$

and

$$\mathcal{F}_2(0,\mu,w)=w-K(a,\mu_0)=w-w_0.$$

Thus, the Fréchet derivative L acting on $(\hat\mu,\hat w)\in\mathbb{R}\times Y$ is given by

$$L(\hat\mu,\hat u)=D_{(\mu,w)}\mathcal{F}(0,\mu_0,w_0)(\hat\mu,\hat w)=(-|\partial\Omega|\hat\mu,\hat w).$$

Since L defines an isomorphism from $\mathbb{R}\times Y$ into itself, the real analytic version of the Implicit Function theorem [**5**] ensures that (3.9) is uniquely solvable with $\mu(t)$ and $w(t,\cdot)$ real analytic in t, and $\lambda=t\mu(t)$ and $u=t^{-\theta}(1+tw(t,\cdot))$ define the unique non-negative solutions to (1.1) for small $\lambda>0$. The remaining assertions in (ii) follow from this representation. □

4. *Behavior for large* λ

In this section, we study the properties of the non-negative non-trivial solutions to (1.1) for large positive λ, assuming $a(x)>0$ in Ω.

Proof of Theorem 1(iii). We use the well-known blow-up technique of Gidas and Spruck (see [**13**]). Assume that (1.2) does not hold. Let $\lambda_n\to+\infty$ and u_n be a sequence of corresponding non-negative non-trivial solutions to (1.1), such that

$$\lambda_n^{2/(1-p)}M_n\to+\infty \tag{4.1}$$

as $n\to+\infty$, where $M_n:=\max_{\overline\Omega}u_n$. Choose points $\tilde x_n\in\partial\Omega$ such that $u_n(\tilde x_n)=M_n$ (notice that the functions u_n are subharmonic), and assume (extracting a subsequence if necessary) that $\tilde x_n\to\tilde x_0\in\partial\Omega$. By means of a translation and a rotation, it can be assumed that $\tilde x_0=0$ while $\{x_N=0\}$ defines the tangent hyperplane to $\partial\Omega$ at such a point. By hypothesis, there exist $R>0$ and $\varphi\in C^{2,\gamma}(B(0,R)\cap\{x_N=0\})$, with $\varphi(0)=0$ and $\nabla\varphi(0)=0$, such that by writing $x\in\mathbb{R}^n$ as $x=(x',x_N)$, where $x'=(x_1,\dots,x_{N-1})$, we have

$$\Omega\cap B(0,R)=\{x:\,x_N>\varphi(x')\}\quad\text{and}\quad\partial\Omega\cap B(0,R)=\{x:\,x_N=\varphi(x')\}.$$

Then, the standard $C^{2,\gamma}$-diffeomorphism $y=h(x)$ given by $y'=x'$, $y_N=x_N-\varphi(x')$ maps $B(0,R)$ onto a neighborhood V of $y=0$ in $\mathbb{R}^N$, $\Omega\cap B(0,R)$ onto $V^+=V\cap\mathbb{R}^N_+$, and $\partial\Omega\cap B(0,R)$ onto $V\cap\partial\mathbb{R}^N_+$. Problem (1.1) in $\Omega\cap B(0,R)$ is transformed into

$$\begin{cases}\Delta u+\displaystyle\sum_{i=1}^{N-1}a_i(y)\frac{\partial^2u}{\partial y_i\partial y_N}+|\nabla\varphi(y')|^2\frac{\partial^2u}{\partial y_N^2}+b(y)\frac{\partial u}{\partial y_N}=a(y)u^p & \text{for } y\in V^+,\\ \nabla u\cdot\nu_1(y)=\lambda u & \text{for } y\in V\cap\partial\mathbb{R}^N_+,\end{cases} \tag{4.2}$$

with

$$a_i=-2\frac{\partial\varphi}{\partial x_i},\qquad b(y)=-\Delta\varphi,\qquad \nu_1=(\nu',-\nu'\nabla\varphi+\nu_n),$$

and $\nu(x)=(\nu'(x),\nu_N(x))$, the outer unit normal on $\partial\Omega$, all such functions being evaluated at $x=h^{-1}(y)$.

On the other hand, setting $\tilde y_n=h(\tilde x_n)$ it follows that $\tilde y_n\to0$ and we can find a positive R_1, not depending on n, such that the translations $w_n(y)=u_n(y+\tilde y_n)$ are all defined in

$B(0,R_1)\cap\overline{\mathbb{R}^N_+}$. Setting $U=B(0,R_1)\cap\mathbb{R}^N_+$ and $U_n=\lambda_n U$ and performing the change

$$v_n(y)=M_n^{-1}u_n(\lambda_n^{-1}y+\tilde{y}_n),$$

we find that the functions v_n define solutions to

$$\begin{cases}\Delta v+\displaystyle\sum_{i=1}^{N-1}a_{i,n}(y)\frac{\partial^2 v}{\partial y_i\partial y_N}+a_{N,n}(y)\frac{\partial^2 v}{\partial y_N^2}+\lambda_n^{-1}b_n(y)\frac{\partial v}{\partial y_N} & \\ \qquad\qquad\qquad\qquad =\lambda_n^{-2}M_n^{p-1}a_n(y)v^p & \text{for } y\in U_n^+,\\ \nabla v\cdot\nu_{1,n}(y)=\lambda v & \text{for } y\in U_n\cap\partial\mathbb{R}^N_+,\end{cases}\tag{4.3}$$

where

$$\begin{aligned}a_{i,n}(y)&=a_i(\lambda_n^{-1}y+\tilde{y}_n)\quad\text{for } 1\leqslant i\leqslant N-1,\\ a_{N,n}(y)&=|\nabla\varphi(\lambda_n^{-1}y'+\tilde{y}'_n)|^2,\\ b_n(y)&=b(\lambda_n^{-1}y+\tilde{y}_n),\\ a_n(y)&=a(\lambda_n^{-1}y+\tilde{y}_n),\\ \nu_{1,n}(y)&=\nu_1(\lambda_n^{-1}y+\tilde{y}_n).\end{aligned}$$

Observe that $U_n\to\mathbb{R}^N_+$ and $\overline{U}_n\to\overline{\mathbb{R}^N_+}$. Since $|v_n|_{\infty,U_n}=1$ for all n, the interior version of the L^p and Schauder estimates imply that (modulo a subsequence) $v_n\to v$ in $C^{2,\beta}(\mathbb{R}^N_+)$ for all $0<\beta<\gamma_1$. Moreover, by employing the 'up to the boundary' version of such estimates in the region $\overline{B(0,R_2)}\cap\mathbb{R}^N_+$, with $R_2>0$ arbitrary and n large, leads to the validity of such convergence in $C^{2,\beta}_{\text{loc}}(\overline{\mathbb{R}^N_+})$. That is why v defines a non-negative solution to

$$\begin{cases}\Delta v=0 & \text{for } x\in\mathbb{R}^N_+,\\ -\dfrac{\partial v}{\partial y_N}=v & \text{for } x\in\partial\mathbb{R}^N_+,\end{cases}\tag{4.4}$$

such that $0\leqslant v\leqslant 1$ and $v(0)=1$. We now show that this is impossible. Consider in fact a C^3-bounded subdomain D of $\mathbb{R}^N_+$ such that

$$\Gamma:=\partial D\cap\partial\mathbb{R}^N_+=\overline{B(0,1)}\cap\partial\mathbb{R}^N_+,$$

and set $\Gamma'=\partial D\cap\mathbb{R}^N_+$. Define as D_n the magnified version $D_n=nD$ of D, and set $\Gamma_n=n\Gamma$ and $\Gamma'_n=n\Gamma'$. According to Theorem 8, the eigenvalue problem

$$\begin{cases}\Delta u=0 & \text{for } x\in D_n,\\ -\dfrac{\partial u}{\partial y_N}=\sigma u & \text{for } x\in\Gamma_n,\\ u=0 & \text{for } x\in\Gamma'_n,\end{cases}\tag{4.5}$$

admits a first eigenvalue $\sigma=\sigma_{1,n}$ with a positive associated eigenfunction

$$\phi_n\in H^1(D_n)\cap W^{2,q}(D_n)\cap C^{2,\gamma}(D_n\cup T)\quad\text{for all } 1<q<4/3,$$

and any closed $T\subset B(0,n)\cap\partial\mathbb{R}^N_+$. Moreover, $\sigma_{1,n}\to 0$ (Remark 4(a)).

Multiplying the equation in (4.4) by ϕ_n, integrating and taking into account Remark 4(b) gives

$$0=(1-\sigma_n)\int_{\Gamma_n}v\phi_n-\int_{\Gamma'_n}v\frac{\partial\phi_n}{\partial\nu}\geqslant(1-\sigma_n)\int_{\Gamma_n}v\phi_n,$$

which is not possible provided n is large.

In conclusion, (4.1) cannot hold, and this proves the theorem. □

As a corollary, we deduce that solutions u_λ to (1.1) develop a dead core

$$\mathcal{O}_\lambda = \{x \in \Omega : u_\lambda(x) = 0\}$$

as λ grows. It turns out that this dead core covers Ω with a speed that can be estimated if $a > 0$ on $\partial\Omega$. For the sake of completeness, we provide next a direct proof of these facts (see also [**3**, **9**]).

Proof of Theorem 1(iv). Let us begin with the case $a > 0$ on $\partial\Omega$. Choose $x \in \Omega$ arbitrary and let $a_0 > 0$ be the infimum of $a(x)$ in Ω. For $y \in B$ (the unit ball), define

$$v(y) = d(x)^{-\alpha} a_0^{-1/(1-p)} u(x + d(x)y),$$

where $\alpha = 2/(1-p)$. Then v satisfies $\Delta v \geqslant v^p$ in B and $v \leqslant \varepsilon := C a_0^{-1/(1-p)} (d(x)\lambda)^{-\alpha}$ on ∂B. It follows that $v \leqslant z$, which is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta z = z^p & \text{in } B, \\ z = \varepsilon & \text{on } \partial B. \end{cases} \tag{4.6}$$

On the other hand, it can be checked directly that the radial function

$$\bar{z}(r) = \begin{cases} 0 & \text{for } r \leqslant \theta, \\ A_N (r-\theta)^\alpha & \text{for } \theta < r < 1, \end{cases}$$

with $r = |x|$, $A_N = [\alpha(\alpha + N - 2)]^{-1/(1-p)}$ (the value A in Theorem 3(iii) corresponds to A_N with $N = 1$), and $\theta = 1 - (\varepsilon/A_N)^{1/\alpha}$, defines a supersolution to (4.6) which equals ε at $r = 1$. Since $\underline{z} = 0$ is a comparable subsolution, it follows by the method of sub and supersolutions (cf. [**2**]) and the uniqueness of solutions to (4.6) that u vanishes in

$$\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geqslant (\varepsilon/A_N)^{1/\alpha} = K\lambda^{-1}\}.$$

The last assertion in (iv) is thus proved.

For the general case one repeats the argument by replacing the above $d(x)$ with $\frac{1}{2}d(x)$ and a_0 with the minimum of a in $\{y \in \Omega : \operatorname{dist}(y, \partial\Omega) \geqslant \frac{1}{2}d(x)\}$ to prove that $\mathcal{O}_\lambda \neq \emptyset$. Then it suffices to observe that $u = 0$ in $\{d(y) \geqslant \delta\}$ provided $u = 0$ on $\{d(y) = \delta\}$. This finishes the proof. □

5. *Multiplicity*

The fact that solutions develop a dead core leads to a failure in uniqueness of non-negative solutions for large λ. Indeed, we prove in this section that for domains Ω with boundary consisting in more than one connected piece, non-negative solutions are not unique.

The proof of this fact relies on constructing solutions whose support for large λ is concentrated near a prefixed connected piece of the boundary. To succeed in this proposal we need to study the following auxiliary version of problem (1.1).

THEOREM 11. *Let $\Omega \subset \mathbb{R}^N$ be a class $C^{2,\gamma}$ bounded domain with $\partial\Omega = \Gamma \cup \Gamma'$, where Γ and Γ' are non-empty and satisfy $\overline{\Gamma} \cap \overline{\Gamma}' = \emptyset$, while $a \in C^\gamma(\overline{\Omega})$, with $a(x) > 0$ for each $x \in \Omega$. Then, the boundary value problem*

$$\begin{cases} \Delta u = a(x) u^p & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{for } x \in \Gamma, \\ u = 0 & \text{for } x \in \Gamma' \end{cases} \tag{5.1}$$

satisfies the following properties.

(i) *Non-trivial and non-negative solutions to* (5.1) *are only possible if* $\lambda > \sigma_1$, *where* σ_1 *is the principal eigenvalue of* (2.4). *In addition,* (5.1) *admits a non-negative non-trivial solution* $u \in C^{2,\gamma_1}(\overline{\Omega})$ *with* $\gamma_1 = \min\{\gamma, p\}$, *for each* $\lambda > \sigma_1$.

(ii) *There exists* $\lambda_1 > \sigma_1$ *such that* (5.1) *admits a* unique *positive solution* $u_\lambda \in C^{2,\gamma_1}(\overline{\Omega})$ *for each* $\sigma_1 < \lambda < \lambda_1$ *where the mapping* $\lambda \to u_\lambda$ *attaining values in* $C^{2,\gamma_1}(\overline{\Omega})$ *is real analytic, and*

$$u_\lambda = \left(\frac{\tilde{\mu}_0}{\lambda - \sigma_1}\right)^{1/(1-p)} \left(\phi_1 + \frac{1}{\tilde{\mu}_0} w(\cdot, \lambda)(\lambda - \sigma_1)\right), \tag{5.2}$$

where ϕ_1 *is a positive normalized eigenfunction associated to* σ_1, $w \in C^{2,\gamma_1}(\overline{\Omega})$, $w_{|\Gamma'} = 0$, $\int_\Omega w\phi_1 = 0$, $w = \tilde{w}_0$ *at* $\lambda = \sigma_1$, *with* $\tilde{\mu}_0 = \left(\int_\Omega a(x)\phi_1^{p+1}\right) / \left(\int_{\Gamma'} \phi_1^2\right)$ *and* $\tilde{w}_0 \in C^{2,\gamma_1}(\overline{\Omega})$ *is the unique solution to the problem*

$$\begin{cases} \Delta w = a(x)\phi_1^p & \text{for } x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = \sigma_1 w + \tilde{\mu}_0 \phi_1 & \text{for } x \in \Gamma, \\ w = 0 & \text{for } x \in \Gamma'. \end{cases}$$

There exist positive constants λ_2 *and* C *such that every non-negative solution* u *satisfies*

$$u \leqslant C\lambda^{-2/(1-p)}, \tag{5.3}$$

for $\lambda \geqslant \lambda_2$. *In addition every non-negative non-trivial solution* u_λ *to* (5.1) *for* $\lambda \geqslant \lambda_2$ *exhibits a dead core* $\mathcal{O}_\lambda = \{u_\lambda(x) = 0\}$. *Moreover, its support concentrates* near Γ *as* $\lambda \to \infty$ *in the sense that* $\{x : \operatorname{dist}(x, \Gamma) \geqslant d_\lambda\} \subset \mathcal{O}_\lambda$ *where* $d_\lambda \to 0+$ *as* $\lambda \to +\infty$. *Again,* $d_\lambda = K/\lambda$, *with* $K > 0$, *if* $a > 0$ *on* Γ.

Proof. Concerning (i) we first observe that the conclusions of Lemma 7 also hold true for weak solutions $u \in H_{\Gamma'}(\Omega) = \{u \in H^1(\Omega) : u_{|\Gamma'} = 0\}$ to (5.1). Accordingly, any non-negative solution $u \neq 0$ does not vanish identically on Γ. Since such a solution satisfies $\int_\Omega |\nabla u|^2 < \lambda \int_\Gamma u^2$ one gets, from (2.5), $\lambda > \sigma_1$.

The existence of a non-negative non-trivial solution for $\lambda > \sigma_1$ follows as in Theorem 1 by minimizing the functional

$$J(u) = \int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2$$

in the set $M = \{u \in H^1_{\Gamma'}(\Omega) : \int_\Omega a(x)|u|^{p+1} = 1\}$. The crucial point now is that J is negative at the infimum only when $\lambda > \sigma_1$ (otherwise, $J \geqslant 0$ and the infimum is attained at $u = 0$).

As for (iii) observe that the proof of the estimate $u_\lambda \leqslant C\lambda^{-2/(1-p)}$ does not require any change compared to the corresponding proof in the case of Theorem 1 since the maximum of u_λ must necessarily be attained on Γ. Therefore and by the same arguments as in § 4, the support of u_λ for large λ is contained in the set $\{\operatorname{dist}(x, \partial\Omega) < d_\lambda\}$ with $d_\lambda \to 0+$. However, we further assert that the support of u_λ is indeed contained in $\{\operatorname{dist}(x, \Gamma) < d_\lambda\}$. In fact, notice that u_λ satisfies $\Delta u \geqslant 0$ in $D = \{0 < \operatorname{dist}(x, \Gamma') < d_\lambda\}$, together with $u_\lambda = 0$ on ∂D and so $u_\lambda = 0$ in D. Hence, the support of u_λ has to be contained in $\{\operatorname{dist}(x, \Gamma) < d_\lambda\}$.

To finish we sketch the proof of (ii). To begin with, it can be shown by using similar reasoning to that in Lemma 10 that any possible sequence (λ_n, u_n) of non-negative non-trivial solutions to (5.1) with $\lambda_n \to \sigma_1+$ satisfies $|u_n|_\infty \to \infty$ and can be written as

$$\lambda_n = \sigma_1 + \tilde{\mu}_n t_n, \qquad u_n = t_n^{1/(1-p)}(\phi_1 + t_n \tilde{w}_n),$$

where $\tilde{\mu}_n \to \tilde{\mu}_0$, $t_n \sim |u_n|_\infty^{p-1} \to 0+$, $\tilde{w}_n \to \tilde{w}_0$ in $C^{2,\beta}(\overline{\Omega})$ for every $0 < \beta < \gamma_1$ where $\tilde{\mu}_0, \tilde{w}_0 \in C^{2,\gamma_1}(\overline{\Omega})$ are given in the statement of the theorem.

On the other hand, for λ close to σ_1 and by writing solutions (λ, u) as $\lambda = \sigma_1 + \mu t$ and $u = t^{-1/(1-p)}(\phi_1 + tw)$ with $t \sim 0$, $\mu \sim \tilde{\mu}_0$ and w close to $\tilde{w}_0$ in the space

$$Z := \left\{ u \in C^{2,\gamma_1}(\overline{\Omega}) :\ u_{|\Gamma'} = 0,\ \int_\Omega u\phi_1 = 0 \right\},$$

problem (5.1) is transformed into

$$\begin{cases} \Delta w = a(x)(\phi_1 + tw)^p & \text{for } x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} - \sigma_1 w = \mu(\phi_1 + tw) & \text{for } x \in \Gamma, \\ w = 0 & \text{for } x \in \Gamma'. \end{cases} \tag{5.4}$$

Taking into account the fact that the linear problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} - \sigma_1 u = g & \text{on } \Gamma, \\ u_{|\Gamma'} = 0 & \text{on } \Gamma', \end{cases}$$

is uniquely solvable in Z for $(f, g) \in X := C^{\gamma_1}(\overline{\Omega}) \times C^{1,\gamma_1}(\overline{\Omega})$ provided

$$\tilde{l}(f, g) := \int_\Omega \phi_1 - \int_{\Gamma'} g\phi_1 = 0$$

(cf. the proof of Lemma 10), we find that the corresponding solution operator $u = K_1(f, g)$ defines an isomorphism $K_1 : \ker \tilde{l} \to Z$. By observing that

$$(f, g) \longrightarrow \left(f, g + \frac{\tilde{l}(f, g)}{\int_\Gamma \phi_1} \right)$$

defines a projection from X onto $\ker \tilde{l}$, we see that solving (5.4) in (t, μ, w) close to $(0, \tilde{\mu}_0, \tilde{w}_0)$ in $\mathbb{R}^2 \times Z$ amounts to solving the equation

$$\mathcal{H}(t, \mu, w) = 0, \tag{5.5}$$

with $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{H}_1(t, \mu, w) = \tilde{l}(F(t, \mu, w), \mu G(t, w))$, $F = a(\phi_1 + tw)^p$, $G = \phi_1 + tw$, and

$$\mathcal{H}_2(t, \mu, w) = w - K_1 \left(F(t, \mu, w), \mu G(t, w) + \frac{\mathcal{H}_1(t, \mu, w)}{\int_\Gamma \phi_1} \right).$$

That (5.5) is uniquely solved in $(-\varepsilon, \varepsilon) \times (\tilde{\mu}_0 - \varepsilon, \tilde{\mu}_0 + \varepsilon) \times B_Z(\tilde{w}_0, \delta)$ for $\varepsilon, \delta > 0$ small follows again from the Implicit Function theorem. As a preliminary checking of regularity observe that F defines a real analytic Nemytskii operator near $(t, \tilde{\mu}_0, \tilde{w}_0)$ in $\mathbb{R}^2 \times Z$ with values in $C^{\gamma_1}(\overline{\Omega})$. In this regard F must be written as

$$F(t, \mu, w) = a(x)\phi_1^p \left(1 + t\frac{w}{\phi_1} \right)^p,$$

and we must keep t small for $w \in B_Z(\tilde{w}_0, \delta)$. By using the facts that both ϕ_1 and w vanish at Γ' and that $\frac{\partial \phi_1}{\partial \nu} < 0$ on Γ', we deduce that the mapping $w \to w/\phi_1$ from Z to $C^1(\overline{\Omega})$ defines a bounded linear operator. This finishes the proof. $\square$

Remark 7. By an argument similar to the one used in the beginning of the proof of Lemma 10, it can be shown that any sequence of solutions $(\lambda_n, u_{\lambda_n})$ to (5.1) is bounded in L^∞ provided λ_n does not accumulate at $\lambda = \tilde{\sigma}_1$. It is also shown by the same means that no sequence can exhibit a subsequence $(\lambda_{n'}, u_{\lambda_{n'}})$ with $\lambda_{n'} \to \lambda \in (\tilde{\sigma}_1, +\infty)$ and $u_{\lambda_{n'}} \to 0$ in $H^1_{\Gamma'}(\Omega^+)$. In particular, (5.1) does not admit $(\lambda, u) = (\lambda, 0)$ as a bifurcation point to non-negative solutions for any $\lambda > \tilde{\sigma}_1$.

The next statement deals with a fixed component Γ_i of $\partial\Omega$. It is an immediate consequence of applying Theorem 11 to Ω under the choice $\Gamma = \Gamma_i$ and $\Gamma' = \partial\Omega \setminus \Gamma_i$ for the boundary conditions.

Corollary 12. *Let Γ_i be any fixed connected component of $\partial\Omega$. Then for large λ problem* (1.1) *has at least one non-negative non-trivial solution u such that $u = 0$ in $\{\mathrm{dist}(x, \Gamma_i) \geqslant d_\lambda\}$, where $d_\lambda \to 0$ as $\lambda \to +\infty$. The rate of convergence d_λ can be chosen as K/λ for certain $K > 0$ if $a > 0$ on Γ_i.*

We can now proceed to the proof of Theorem 2.

Proof of Theorem 2. If $\Gamma_1, \dots, \Gamma_k$ are the connected components of $\partial\Omega$ then Corollary 12 implies the existence of k non-negative non-trivial solutions $u_{\lambda,i}$ whose support is localized near Γ_i for large λ. The supports will indeed be disjoint for large λ. This allows us to conclude that every sum of the form $u_{\lambda,i_1} + \dots + u_{\lambda,i_r}$ is again a solution to (1.1). Since there are $2^k - 1$ such sums, we have shown that there exist at least $2^k - 1$ different non-negative non-trivial solutions to (1.1) for large λ. □

6. *The case of the ball*

In this section we consider a special case of problem (1.1) by analyzing its behavior in the unit ball B of $\mathbb{R}^N$. To simplify the exposition we focus our interest on the autonomous case represented by the coefficient $a(x) = 1$. We begin our treatment by searching only for non-negative radially symmetric solutions $u = u(r)$, $r = |x|$. They satisfy

$$\begin{cases} (r^{N-1}u')' = r^{N-1}u^p & \text{for } 0 \leqslant r < 1, \\ u'(0) = 0, \quad u'(1) = \lambda u(1). \end{cases} \tag{6.1}$$

The scaling $v(r) = \lambda^\alpha u(\lambda^{-1}r)$, with $\alpha = 2/(1-p)$, transforms this problem into

$$\begin{cases} (r^{N-1}v')' = r^{N-1}v^p & \text{for } 0 \leqslant r < \lambda, \\ v'(0) = 0, \quad v'(\lambda) = v(\lambda). \end{cases} \tag{6.2}$$

Modifying the proof of point (i) in Theorem 1 by restricting both J and M to the class of radial functions in $H^1(B)$, one can show the existence of a non-trivial non-negative radial solution to (1.1) for all $\lambda > 0$, and hence the equivalent assertion for problem (6.1), which is point (i) in Theorem 3. On the other hand, it has been already shown (Theorem 1(iv)) that all non-negative solutions exhibit a dead core for large λ which completely fills B as $\lambda \to +\infty$. This implies the existence of $d = d(\lambda) > 0$, where $\lambda - K \leqslant d < \lambda$ and so $d \to \infty$ as $\lambda \to \infty$, such that any non-negative and non-trivial solution v to (6.2) satisfies $v = 0$ in $[0, d]$ and $v > 0$ in $(d, \lambda]$. Therefore, setting $w(r) = v(d + r)$, we see that w defines a solution to

$$\begin{cases} ((r+d)^{N-1}w')' = (r+d)^{N-1}w^p & \text{for } 0 \leqslant r < \lambda - d, \\ w(0) = 0, \quad w'(0) = 0, \quad w'(\lambda - d) = w(\lambda - d). \end{cases} \tag{6.3}$$

Such a solution is non-trivial in the sense that $w(r) > 0$ for $r \in (0, \lambda - d)$ (observe that solutions to (2.6) are *non-trivial*; see § 2).

Exploiting these remarks we now show items (ii) and (iii) in Theorem 3 (compare with the proof of [**12**, Theorem 3.4]).

Proof of Theorem 3(ii) *and* (iii). To prove (ii) notice that the uniqueness assertion in Theorem 1(ii) furnishes a unique radial positive solution u_λ for $0 < \lambda < \lambda_0$ satisfying the properties stated there.

To show the uniqueness for large λ, consider the Cauchy problem

$$\begin{cases} ((r+d)^{N-1}w')' = (r+d)^{N-1}w^p & \text{in } (0,\delta), \\ w(0)=0, \quad w'(0)=0, \end{cases} \tag{6.4}$$

which, by Theorem 9, has a unique non-trivial solution $w(r,d)$ in $[0,+\infty)$ for every $d \geqslant 0$. Moreover, as $d \to +\infty$, w converges in $C^1_{\mathrm{loc}}[0,+\infty)$ to $w_0(r) = Ar^\alpha$, where A and α are as in the statement of the theorem.

By the preceding discussion, showing the uniqueness amounts to proving the existence as $d \to \infty$ of a unique positive zero $r = T$ of the equation,

$$w'(r,d) - w(r,d) = 0, \tag{6.5}$$

such that $T = O(1)$ while $T + d = \lambda$ is uniquely solvable in d as $d \to \infty$.

The solvability with uniqueness of (6.5) for d large follows from the fact that the limit equation $w_0'(r) - w_0(r) = 0$ has $r = \alpha$ as a unique simple zero. In fact observe that $w_0''(\alpha) - w_0'(\alpha) = -A\alpha^{\alpha-1}$. Thus (6.5) admits a root $T(d)$ such that $T(d) \to \alpha$ as $d \to +\infty$.

We now claim that $T(d)$ is a C^1-function for large d and satisfies $T'(d) \to 0$ as $d \to +\infty$. This is a consequence of the Implicit Function theorem; we see this by first noticing that $w(r,d)$ is C^1 in both variables (Theorem 9), while in addition

$$T'(d) = \frac{w_d(T(d),d) - w_d'(T(d),d)}{w''(T(d),d) - w'(T(d),d)}$$

(the subscript d denotes derivation with respect to d and $'$ stands for the derivative with respect to r). As already seen, the denominator is negative for large d since it converges to $w_0''(\alpha) - w_0'(\alpha) = -A\alpha^{\alpha-1}$, while the numerator tends to zero thanks to (2.7) in Theorem 9. This proves the claim.

Finally, notice that solutions to (6.3) arise whenever there is a solution to the equation $d + T(d) = \lambda$. Since $d + T(d)$ is increasing for large d, it follows that for large λ there is a unique $d = d(\lambda)$ so that $w(r,d)$ is a solution to (6.3). The unique solution to (6.1) for large λ is then obtained by setting

$$u_\lambda(r) := \begin{cases} 0 & \text{if } 0 \leqslant r \leqslant d/\lambda, \\ \lambda^{-\alpha} w(\lambda r - d) & \text{if } r > d/\lambda. \end{cases} \tag{6.6}$$

This proves part (ii).

Part (iii) follows at once on noticing that $w(T(d),d) \to w_0(\alpha) = A\alpha^\alpha$ and

$$r(\lambda) = d(\lambda)/\lambda = 1 - T(d(\lambda))/\lambda \sim 1 - \alpha/\lambda. \qquad \square$$

Corollary 13. *Let u_λ be the unique non-negative non-trivial radial solution to* (1.1) *for large λ. Then*

$$\int_B u_\lambda^{p+1} \sim \omega_N A^{p+1} \frac{\alpha^{\alpha(p+1)+1}}{\alpha(p+1)+1} \lambda^{-\alpha(p+1)-1}, \quad \text{as } \lambda \to +\infty,$$

where ω_N is the surface measure of the unit ball of $\mathbb{R}^N$, and A and α are given in Theorem 3.

Proof. According to (6.6), we have

$$\begin{aligned}\int_B u_\lambda^{p+1} &= \omega_N \lambda^{-\alpha(p+1)} \int_{d/\lambda} r^{N-1} w(\lambda r - d)^{p+1}\, dr \\ &= \omega_N \lambda^{-\alpha(p+1)-1} \int_0^{\lambda-d} \left(\frac{s+d}{\lambda}\right)^{N-1} w(s)^{p+1}\, ds \\ &\sim \omega_N \lambda^{-\alpha(p+1)-1} \int_0^{\alpha} w_0(s)^{p+1}\, ds \\ &= \omega_N A^{p+1} \frac{\alpha^{\alpha(p+1)+1}}{\alpha(p+1)+1} \lambda^{-\alpha(p+1)-1},\end{aligned}$$

as $\lambda \to \infty$. □

Of course one could ask the question whether there are non-radial solutions, or if the radial one is the unique one. Since ∂B is connected, we can not obtain multiplicity as in § 5. However, we prove next that besides the radial solution there is indeed another non-trivial non-negative and non-radial solution for large λ.

Proof of Theorem 3(iv). We show that the solution constructed in Theorem 1(i) is not the radial one. Denote, as in that theorem,

$$J(u) = \int_B |\nabla u|^2 - \lambda \int_{\partial B} u^2, \quad \text{where } u \in M = \left\{ u \in H^1(B) : \int_B |u|^{p+1} = 1 \right\}.$$

For the radial solution u_λ we have

$$J(u_\lambda) = -\int_B u_\lambda^{p+1},$$

so that $v_\lambda = u_\lambda / |u_\lambda|_{L^{p+1}} \in M$ satisfies $J(v_\lambda) = -|u_\lambda|_{L^{p+1}}^{-(1-p)/(1+p)}$. Then by Corollary 13, there exists a positive constant C so that

$$J(v_\lambda) \sim -C\lambda^{(p+3)/(p+1)}, \quad \text{as } \lambda \to +\infty. \tag{6.7}$$

We next use (6.7) to prove that $J(v_\lambda)$ is not the minimum of J in M for large λ. For this aim, we construct a family of functions $\psi_\lambda \in M$ so that $J(\psi_\lambda) < J(v_\lambda)$ for large λ.

We first claim that a large radius $R > 0$ and a function $\psi \in C_0^\infty(B_R)$, where $B_R = B(0, R)$, can be found so that

$$\int_{B_R^+} |\nabla \psi|^2 - \int_{\Gamma_R} \psi^2 < 0, \tag{6.8}$$

where $B_R^+ = B_R \cap \mathbb{R}_+^N$ and $\Gamma_R = B_R \cap \partial \mathbb{R}_+^N$.

Assuming the existence of such a ψ, set

$$\psi_\lambda(x) = C(\lambda)\psi(\lambda(x + e_N)) \in C_0^\infty(B(-e_N, \lambda^{-1}R)),$$

which belongs to M provided $C(\lambda) = K\lambda^{N/(p+1)}$, where $K^{-1} = |\psi|_{p+1, B_R}$. With this choice of ψ_λ, we have

$$\begin{aligned}J(\psi_\lambda) &= K^2 \lambda^{2N/(p+1)+2-N} \left(\int_{B_R \cap B(\lambda e_N, \lambda)} |\nabla \psi|^2 - \int_{B_R \cap \partial B(\lambda e_N, \lambda)} \psi^2 \right) \\ &\sim K^2 \lambda^{(1-p)N/(1+p)+2} \left(\int_{B_R^+} |\nabla \psi|^2 - \int_{\Gamma_R} \psi^2 \right).\end{aligned}$$

Since $(1-p)N/(1+p)+2 > (p+3)(p+1)$, we obtain by virtue of (6.7) and (6.8), that $J(\psi_\lambda) < J(v_\lambda)$ for large λ. As remarked before, this entails the existence of a non-trivial non-negative and non-radial solution to (1.1).

To show the claim let $\phi \in H^1(B_R^+)$ be a positive eigenfunction to

$$\begin{cases} \Delta u = 0 & \text{for } x \in B_R^+, \\ -\dfrac{\partial u}{\partial x_N} = \sigma u & \text{for } x \in \Gamma_R, \\ u = 0 & \text{for } x \in \Gamma_R', \end{cases} \tag{6.9}$$

corresponding to the principal eigenvalue $\sigma = \sigma_1(R)$ in [**11**, Theorem 6 and Remark 8]. Provided R is big enough so that $\sigma_1(R) < 1$ (see Remark 4(a)), ϕ satisfies

$$\int_{B_R^+} |\nabla \phi|^2 - \int_{\Gamma_R} \phi^2 = (\sigma_1(R) - 1) \int_{\Gamma_R} \phi^2 < 0.$$

The required function ψ can be now obtained from ϕ by regularization. This concludes the proof. □

7. *Vanishing weights*

In this final section, we give the proof of Theorems 4, 5 and 6, considering in problem (1.1) a weight $a(x)$ that vanishes in a subdomain of Ω.

Proof of Theorem 4. We only sketch the proof of part (i), the remaining points being proved in exactly the same way as in Theorem 1(ii), (iii) and (iv). If u is a non-negative solution to (1.1) with $\lambda \leqslant 0$ (which, due to Lemma 7, can be supposed in $C^{2,\gamma_1}(\overline{\Omega})$), we deduce that

$$\int_\Omega a(x) u^p = 0,$$

and so $u = 0$ in $\Omega \setminus \Omega_0$. Thus u satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_0, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases}$$

and it follows from the maximum principle that $u = 0$ in Ω_0 as well. Hence, $u \equiv 0$.

Assume now that $0 < \lambda < \sigma_1$. If the functional

$$J(u) = \int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2$$

were not coercive in $M := \left\{u \in H^1(\Omega) : \int_\Omega a(x)|u|^{p+1} = 1\right\}$, then we would obtain (proceeding as in the proof of Theorem 1(i)) a function $v \in H^1(\Omega)$ with $|v|_{L^2(\partial\Omega)} = 1$ and satisfying the relations

$$\int_\Omega |\nabla v|^2 \leqslant \lambda, \qquad \int_\Omega a(x)|v|^{p+1} = 0.$$

The last equation implies that $v = 0$ in $\Omega \setminus \Omega_0$ (in particular on Γ_2). If $\sigma_1 = +\infty$, it follows that $v = 0$ on $\partial\Omega$ which is impossible. If, on the contrary, $\sigma_1 < \infty$ then (2.5) yields $\sigma_1 \leqslant \lambda$, also a contradiction. Hence J is coercive and there exists a non-negative non-trivial solution to (1.1). □

Lemma 14. *Assume $\sigma_1 < +\infty$, and let $u \in C^{2,\gamma}(\overline{\Omega})$ be a non-negative solution to* (1.1) *with $\lambda \geqslant \sigma_1$. Then $u \equiv 0$ in Ω_0.*

Proof. Multiplying (1.1) by the eigenfunction ϕ_1 associated to σ_1 and integrating in Ω_0 (notice that $\phi_1 \in C^{2,\gamma}(\overline{\Omega})$ and $u \in C^{2,\gamma_1}(\overline{\Omega})$), we obtain

$$(\lambda - \sigma_1)\int_{\Gamma_1} u\phi_1 = \int_{\Gamma_2} u\frac{\partial \phi_1}{\partial \nu}. \tag{7.1}$$

If $\lambda > \sigma_1$, we obtain directly $u = 0$ on $\Gamma_1 \cup \Gamma_2 = \partial\Omega_0$, and since $\Delta u = 0$ in Ω_0, it follows that $u \equiv 0$ in Ω_0.

If $\lambda = \sigma_1$, then we have, from (7.1), $u = 0$ on Γ_2. Thus, u is an eigenfunction associated to σ_1, and the simplicity of σ_1 (see [**11**, Theorem 6]) implies that $u = c\phi$ for some non-negative constant c. If $c > 0$, then $\frac{\partial u}{\partial \nu} < 0$ on Γ_2, which would imply that u changes sign in Ω. Thus $c = 0$ and we arrive again at $u \equiv 0$ in Ω_0. This completes the proof. □

Proof of Theorem 5. To show (i), recall that $\tilde{\sigma}_1 = +\infty$ is equivalent to $\Gamma_1 = \partial\Omega$. Since by Lemma 14 we have $u = 0$ on Γ_1 for every non-negative solution u with $\lambda \geqslant \sigma_1$ and u is subharmonic, we arrive at $u = 0$ in Ω.

As for (ii), suppose u is a non-negative non-trivial solution corresponding to $\lambda \geqslant \sigma_1$. By Lemma 14, $u \equiv 0$ in Ω_0 which implies $u = 0$ on $\Gamma_1 \cup \Gamma_2$. Since $\partial\Omega^+ \cap \Omega = \Gamma_2$, then u satisfies

$$\begin{cases} \Delta u = a(x)u^p & \text{for } x \in \Omega_i^+, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{for } x \in \partial\Omega_i^+ \cap \partial\Omega, \\ u = 0 & \text{for } x \in \partial\Omega_i^+ \cap \Omega, \end{cases} \tag{7.2}$$

for each component Ω_i^+ having $\partial\Omega_i^+ \cap \partial\Omega \neq \emptyset$. Observe that $u = 0$ in the remaining components $\Omega_j^+ \subset\subset \Omega$ since $u = 0$ on $\partial\Omega_j^+$ and is subharmonic there. Therefore, provided $u \not\equiv 0$, u must define a non-trivial solution of some of the problems (7.2). By Theorem 11, this means that $\lambda > \tilde{\sigma}_{1,i} \geqslant \tilde{\sigma}_1$.

Finally, according to (iii) of Theorem 11, each of the problems (7.2) defines for $\lambda > \tilde{\sigma}_1$ large a non-trivial solution to (1.1). These solutions and a combination of them (in the spirit of the proof of Theorem 2) ensure the validity of (iii). □

Proof of Theorem 6. Observe that under the assumptions of the theorem $u = 0$ is the only solution to (1.1) for $\lambda = \sigma_1$.

Now let $\lambda_n \to \sigma_1-$ with u_n a corresponding non-negative non-trivial solution to (1.1) at $\lambda = \lambda_n$. We claim that $t_n := |u_n|_\infty$ is bounded. If not, passing to a subsequence we can assume that $t_n \to +\infty$. Denoting $v_n = u_n/t_n$, we have

$$\begin{cases} \Delta v_n = a(x)v_n^p|u_n|^{p-1} & \text{in } \Omega, \\ \dfrac{\partial v_n}{\partial \nu} = \lambda_n v_n & \text{on } \partial\Omega. \end{cases}$$

It is standard procedure to obtain the fact that a subsequence of $\{v_n\}$ converges in $C^2(\overline{\Omega})$ to a function v which satisfies $|v|_\infty = 1$ and

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \sigma_1 v & \text{on } \partial\Omega. \end{cases} \tag{7.3}$$

By integrating (7.3) directly, we find that $v = 0$, and this contradicts $|v|_\infty = 1$. Thus t_n is bounded, and we can obtain, as before, a subsequence $u_n \to u$ in $C^{2,\beta}(\overline{\Omega})$, with $0 < \beta < \gamma_1$, where u is a solution to (1.1) with $\lambda = \sigma_1$. Therefore $u = 0$, and this proves the convergence $u_\lambda \to 0$.

Regarding the dead core formation, it follows, from the fact that $\sup_\Omega u_\lambda \to 0$ as $\lambda \to \sigma_1-$, that a non-empty interior region $\{u_\lambda = 0\}$ is generated in Ω^+ as $\lambda \to \sigma_1-$ (cf. the proof of

Theorem 1(iv)). However, such a region can never reach Γ_2 or Ω_0 for λ close to σ_1. In fact, $u_\lambda(x_0)=0$ at $x_0\in\Gamma_2$ implies $u_\lambda=0$ in Ω_0. Otherwise, since u_λ is harmonic in Ω_0 we get $\frac{\partial u_\lambda}{\partial\nu}(x_0)<0$, where ν is the outward unit normal to Ω_0 at x_0, which contradicts the non-negativeness of u_λ. Hence $u_\lambda=0$ in Ω_0 (notice that such behavior is immediately achieved if $u_\lambda=0$ somewhere in Ω_0). This means that we get a family u_λ of non-trivial solutions to the mixed problem (5.1) in Ω^+ such that $u_\lambda\to 0$ as $\lambda\to\sigma_1-$. Observe that the existence of such a family is forbidden if $\sigma_1\leqslant\tilde\sigma_1$ since such behavior is not possible if $\tilde\sigma_1<\sigma_1$, as observed in Remark 7. This shows that $\mathcal{O}_\lambda\subset\Omega^+$ for $\lambda\to\sigma_1-$. The remaining assertions follow in the same way as in the case of Theorem 1. □

Remark 8. In order to illustrate with an example the relative values of σ_1 and $\tilde\sigma_1$ consider the annulus $\Omega=\{R_1<|x|<R_2\}$, with $0<R_1<R_2$ fixed. For $R_1<R<R_2$ variable, set $\Omega^+=\{R_1<|x|<R\}$ and $\Omega_0=\{R<|x|<R_2\}$, and so $\Gamma_1=\{|x|=R_2\}$, $\Gamma_2=\{|x|=R\}$, and $\Gamma^+=\partial\Omega\setminus\Gamma_1=\{|x|=R_1\}$. The principal eigenvalues σ_1 and $\tilde\sigma_1$ (see (1.3) and (1.4)) must in this case be associated to radial (harmonic) eigenfunctions and hence are explicitly given by

$$\sigma_1=\frac{(N-2)R^{N-2}}{R_2(R_2^{N-2}-R^{N-2})},\qquad \tilde\sigma_1=\frac{(N-2)R^{N-2}}{R_1(R^{N-2}-R_1^{N-2})}. \tag{7.4}$$

Observe that $\sigma_1=\tilde\sigma_1$ at $R=R^*$, where $R^*/R_1=(\zeta^{N-1}+1)/(\zeta+1)$ and $\zeta=R_2/R_1$, that $\sigma_1<\tilde\sigma_1$ for $R_1<R<R^*$, and that $\sigma_1>\tilde\sigma_1$ if $R^*<R<R_2$. This shows that it is possible to have both situations $\sigma_1\leqslant\tilde\sigma_1$ and $\sigma_1>\tilde\sigma_1$.

Let us now construct examples showing the behaviors announced in Remark 1(b), namely a situation where $\tilde\sigma_1<\sigma_1$ and no solutions exist for either $\lambda>\sigma_1$ or $\lambda\sim\sigma_1$, or another situation where solutions exist for $\sigma_1-\varepsilon<\lambda<\sigma_1$ for a certain $\varepsilon>0$. In what follows, we have in mind the fact that R is fixed, while R_2 is going to vary. Choose $a\in C^\gamma(\mathbb{R}^N)$ such that $a>0$ for $|x|<R$, and $a=0$ in $|x|\geqslant R$, and consider the problem

$$\begin{cases}\Delta u=a(x)u^p & \text{for } x\in\Omega^+,\\ \dfrac{\partial u}{\partial\nu}=\lambda u & \text{for } x\in\Gamma^+,\\ u=0 & \text{for } x\in\Gamma_2.\end{cases} \tag{7.5}$$

Theorem 11 provides the existence of $\tilde\sigma_1<\lambda^*\leqslant\lambda_1$ (see (ii)) such that the unique positive solution u_λ to (7.5) satisfies $\frac{\partial u_\lambda}{\partial\nu}<0$ on Γ_2 (here ν is the outward unit normal to Ω^+ on Γ_2) for each $\tilde\sigma_1<\lambda<\lambda^*$.

Now in order to obtain an example of the first situation, notice that (7.4) allows one to choose R_2 such that $\tilde\sigma_1<\sigma_1<\lambda^*$. This means that problem (1.1) cannot exhibit a non-negative non-trivial solution u at least for any $\sigma_1\leqslant\lambda\leqslant\lambda^*$. In fact, such a possible solution u should vanish in $\overline{\Omega}_0$ (Lemma 14) thus providing a non-negative non-trivial solution to (7.5) with $\frac{\partial u_\lambda}{\partial\nu}=0$ on Γ_2 which is not possible in such a range for λ.

To achieve an example of the second situation it suffices to choose R_2 so that $\sigma_1>\lambda_2$ with λ_2 as in Theorem 11(ii). This completes the announced constructions.

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