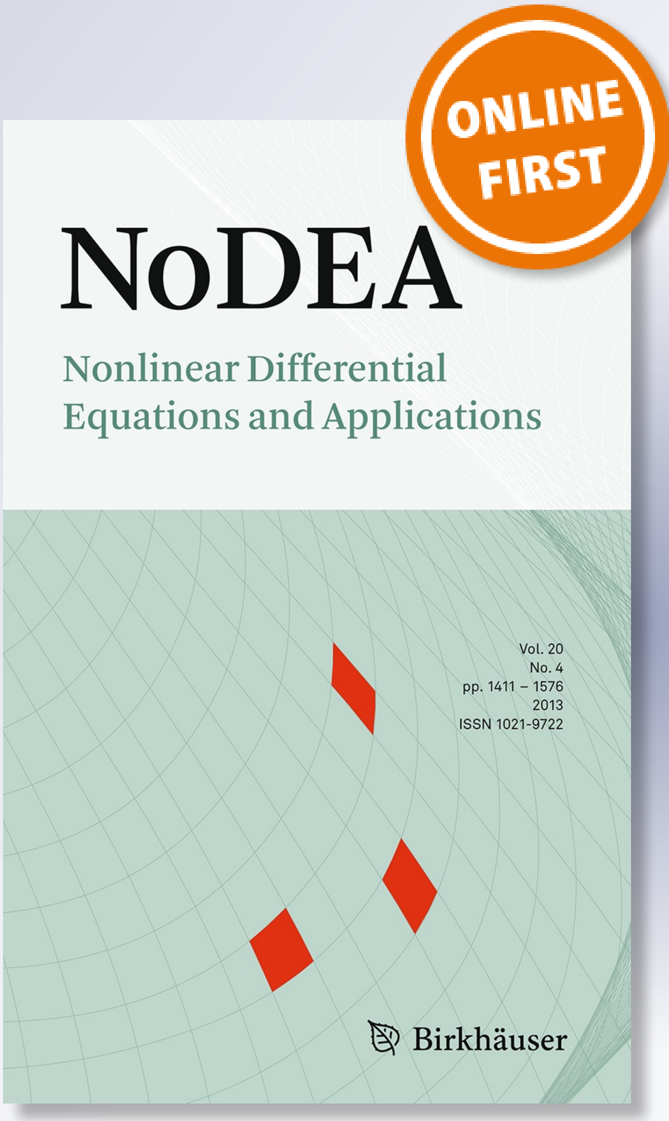
Multiplicity of solutions to a nonlinear elliptic problem with nonlinear boundary conditions

Jorge García-Melián, Julio D. Rossi & José C. Sabina de Lis

Nonlinear Differential Equations and Applications NoDEA

ISSN 1021-9722

Nonlinear Differer. Equ. Appl.
DOI 10.1007/s00030-013-0248-8



Nonlinear Differ. Equ. Appl.

DOI 10.1007/s00030-013-0248-8

Nonlinear Differential Equations and Applications NoDEA

Multiplicity of solutions to a nonlinear elliptic problem with nonlinear boundary conditions

Jorge García-Melián, Julio D. Rossi and José C. Sabina de Lis

Abstract. We study the problem

$$\begin{cases} \Delta_p u = |u|^{q-2}u, & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, ν is the outward unit normal at $\partial\Omega$ and $\lambda > 0$ is regarded as a bifurcation parameter. When $p = 2$ and in the superlinear regime $q > 2$, we show existence of n nontrivial solutions for all $\lambda > \lambda_n$, λ_n being the n-th Steklov eigenvalue. It is proved in addition that bifurcation from the trivial solution takes place at all λ_n's. Similar results are obtained in the sublinear case $1 < q < 2$. In this case, bifurcation from infinity takes place in those λ_n with odd multiplicity. Partial extensions of these features are shown in the nonlinear diffusion case $p \neq 2$ and related problems under spatially heterogeneous reactions are also addressed.

Mathematics Subject Classification (2010). 35J66, 35J70, 35J20, 35B32.

Keywords. Index theory, Steklov problem, variational methods, bifurcation theory.

1. Introduction

Since the seventies, a great deal of attention has been focused on the study of nonlinear boundary value problems involving the so-called reaction diffusion equations. In those problems, the occurrence of a bifurcation parameter λ which often exerts its influence on the reaction term is common (see for instance [22] for pioneering review on the subject). However, much less studied

Julio D. Rossi is on leave from Departamento de Matemática, FCEyN UBA, Ciudad Universitaria, Pab 1 (1428), Buenos Aires, Argentina.

is the case when the parameter λ appears in the boundary condition (see, for example, [5–7, 12, 13], and references therein).

In this paper we continue the analysis of the reaction–diffusion problem

$$\begin{cases} \Delta u = \varphi_q(u), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, ν is the outward unit normal on $\partial\Omega$, $\varphi_q(u) = |u|^{q-2}u$ with $q > 1$ and λ is a bifurcation parameter. In fact, (1.1) was studied in the case $q > 2$ as the prototype of a class of problems under a superlinear reaction in [12], meanwhile the sublinear regime $1 < q < 2$ for the exponent together with a broader class of problems related to (1.1) was treated in detail in [13]. In those works only nonnegative solutions were considered.

The present work goes a step further and is concerned with the study of two-signed solutions to (1.1). Moreover, it also deals with the nonlinear diffusion version of (1.1)

$$\begin{cases} \Delta_p u = \varphi_q(u), & x \in \Omega, \\ \mathcal{B}_p(u) = \lambda \varphi_p(u), & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p u = \operatorname{div}\,(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $p > 1$ and $\mathcal{B}_p$ stands for the boundary operator associated with the flux, that is,

$$\mathcal{B}_p(u) = |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}.$$

In order to introduce our results we distinguish in some occasions between the linear ($p = 2$) and the nonlinear ($p \neq 2$) diffusion problems (1.1) and (1.2), respectively. As should be expected, statements for (1.2) will be substantially weaker than those for (1.1). In addition, we address separately the superlinear regime $q > p > 1$ and the sublinear one $1 < q < p$ [notice that the homogeneity of the reaction term in the problem in both (1.1) and (1.2) is $q - 1$].

Here we deal with weak solutions u to (1.2). As it is customary in the theory, we say that $u \in W^{1,p}(\Omega) \cap L^q(\Omega)$ solves (1.2) provided the equality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \psi - \lambda \int_{\partial\Omega} \varphi_p(u)\psi + \int_\Omega \varphi_q(u)\psi = 0, \tag{1.3}$$

holds for all $\psi \in W^{1,p}(\Omega)$. Thus, such solutions u are given as critical points of the associated energy functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p + \frac{1}{q}\int_\Omega |u|^q. \tag{1.4}$$

However, some caution is needed to deal with the last integral. Indeed, we want J not only to be merely well defined but also to be of class C^1 in $W^{1,p}(\Omega)$. Since we are facing the regime $q > p$ this creates a difficulty in the case $1 < p < N$ if q is in the supercritical range $q > p^* := \frac{Np}{N-p}$. To overcome the difficulty and in order to handle the full regime $q > p$ it will be shown that solutions u to (1.2) are bounded in Ω [see Proposition 10 and Remarks 10 (b)]. This not only implies that weak solutions u become $C^{1,\beta}$ for some $0<\beta<1$ but it also allows us to construct a suitable C^1 truncation $\widetilde{J}$ of the functional J which possesses the same critical points as J. Thus, variational results on the truncated functional $\widetilde{J}$ provide solutions to (1.2).

We begin now with the statement of our results for the case $q>p$. The first one essentially reviews basic features on (1.2) (some of them formerly introduced in [14,15]).

Theorem 1. *Let $\Omega\subset\mathbb{R}^N$ be a class $C^{1,\alpha}$ bounded domain for a certain $0<\alpha<1$ and assume that the exponent q satisfies $q>p$. Then problem* (1.2) *satisfies the following properties.*

i) *Nontrivial solutions to* (1.2) *are only possible when $\lambda>0$. Moreover, for each $\lambda>0$ a unique positive solution $u=u_\lambda\in C^{1,\beta}(\overline{\Omega})$ exists so that $u=\pm u_\lambda$ are the unique one-signed solutions to* (1.2).

ii) *u_λ is increasing and continuous with respect to λ and bifurcates from zero at $\lambda=0$. More precisely*

$$u_\lambda=(|\partial\Omega|/|\Omega|)^{\frac{1}{q-p}}\lambda^{\frac{1}{q-p}}(1+o(1)),$$

as $\lambda\to 0$ in $C^{1,\beta}(\overline{\Omega})$. Here $|\cdot|$ stands for the Lebesgue measure in the appropriate dimension.

iii) *$u_\lambda\to U$ as $\lambda\to\infty$ in $C^{1,\beta}(\Omega)$, where U stands for the unique solution to the singular problem*

$$\begin{cases}\Delta_p U=\varphi_q(U), & x\in\Omega,\\ U=\infty, & x\in\partial\Omega.\end{cases}\tag{1.5}$$

Remark 1. It should be observed that no restriction on the size of $q>p$ is imposed in Theorem 1 and remaining statements on the superlinear case.

Our next results concern the linear diffusion problem (1.1) in the superlinear regime $q>2$. The Steklov eigenvalues λ_n are involved in their statements. A brief description on the Steklov eigenvalue problem

$$\begin{cases}\Delta u=0, & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu}=\lambda u, & x\in\partial\Omega,\end{cases}$$

is contained in Sect. 2.2. A first result provides existence of nontrivial solutions for all values of λ greater than λ_n.

Theorem 2. *Assume $\Omega\subset\mathbb{R}^N$ is a $C^{1,\alpha}$ bounded domain and $q>2$. Then, for every Steklov eigenvalue λ_n and each $\lambda>\lambda_n$, problem* (1.1) *possesses at least n pairs $\pm u_{k,n}(\lambda)$, $1\le k\le n$, of nontrivial solutions.*

Remark 2. In view of Theorem 1 we can ensure that $n-1$ pairs of the solutions obtained in Theorem 2 are sign changing.

In Theorem 2 nontrivial solutions are obtained by a variational argument involving index theory ([3] and Sect. 2.1). This argument is of global nature since existence of nontrivial solutions is obtained for all $\lambda>\lambda_n$. However, nothing is said on the nature and possible onset of such solutions. The following result strongly suggests that "bifurcation" from $(\lambda_n, 0)$ is the most likely mechanism generating such solutions.

As a matter of terminology, it is said that $(\bar{\lambda}, 0)$ is a bifurcation point of solutions to either problems (1.1) or (1.2) if sequences $\bar{\lambda}_k \to \bar{\lambda}$ and $u_k \in C^{1,\beta}(\overline{\Omega})$ exist such that u_k is a *nontrivial* solution corresponding to $\lambda = \bar{\lambda}_k$ and $u_k \to 0$ in $C^{1,\beta}(\overline{\Omega})$ (the u_k's are referred to as "bifurcated" solutions).

Theorem 3. *Under the hypotheses of Theorem 2 on Ω and q the following properties hold:*

i) *For* every *Steklov eigenvalue λ_n, $(\lambda, u) = (\lambda_n, 0)$ defines a bifurcation point of solutions to (1.1). Moreover, bifurcated solutions from $(\lambda_n, 0)$ occur only near and to the right of λ_n.*
ii) *For each $n \in \mathbb{N}$ there exists some $\delta_0 = \delta_0(n)$ such that for every $\lambda \in (\lambda_n, \lambda_n + \delta_0)$ problem (1.1) admits at least m pairs $\pm u_{k,n}(\lambda)$, $1 \le k \le m$, of nontrivial solutions satisfying $u_{k,n}(\lambda) \to 0$ as $\lambda \to \lambda_n$, where $m = m(n)$ is the multiplicity of λ_n.*
iii) *In case that λ_n is simple, bifurcated solutions in* ii) *consist exactly in a single pair $\pm u_{1,n}(\lambda)$. Moreover, $u_{1,n}(\lambda)$ is a continuous function of $\lambda \in (\lambda_n, \lambda_n + \delta_0)$ such that*

$$u_{1,n}(\lambda) = E_n^{\frac{1}{q-2}} (\lambda - \lambda_n)^{\frac{1}{q-2}} (\phi_{1,n} + o(1)), \tag{1.6}$$

in $C^{1,\beta}(\overline{\Omega})$ as $\lambda \to \lambda_n+$, where

$$E_n = \frac{\int_{\partial\Omega} \phi_{1,n}^2}{\int_\Omega |\phi_{1,n}|^q}$$

with $\phi_{1,n}$ the associated eigenfunction, modulus a multiplicative ± 1, satisfying

$$\int_\Omega |\nabla \phi_{1,n}|^2 + \int_{\partial\Omega} \phi_{1,n}^2 = 1.$$

Remark 3. It is expected that for a "generic" domain Ω every Steklov eigenvalue is simple (as is just the case for the other classical eigenvalue problems). Thus, in this special case one can think of the solutions in Theorem 2 as arising in pairs every time λ crosses a (simple) eigenvalue λ_n. However, to properly put this assertion on firm ground one first needs to show that the bifurcated branches $u_n^{\pm}(\lambda)$ can be continued for all $\lambda > \lambda_n$ without collapsing to $u = 0$. This is a nontrivial issue that is not going to be pursued here.

We return now to the nonlinear diffusion problem (1.2) in the superlinear range $q > p$, and state a weak version of Theorem 2. We deal with the $p-$Laplacian version (2.2) of the Steklov eigenvalue problem:

$$\begin{cases} \Delta_p u = 0, & x \in \Omega, \\ \mathcal{B}_p(u) = \lambda\varphi_p(u), & x \in \partial\Omega. \end{cases}$$

In the next statement $\lambda_{n,p}$ stands for a suitable sequence of Steklov "p-eigenvalues" which exhibits the feature of generating a corresponding *linearly independent* family of eigenfunctions $\{\phi_{n,p}\}$ in the space $L^p(\partial\Omega)$. Existence of such family $\lambda_{n,p}$ and further remarks on this nonlinear eigenvalue problem are contained in Sect. 2.2 below.

Theorem 4. *Suppose $\Omega \subset \mathbb{R}^N$ is a $C^{1,\alpha}$ bounded domain and assume that the exponents verify $q > p > 1$. Then, the following properties hold true.*

i) *For a fixed λ, the set of all possible solutions to (1.2) constitutes a compact set in $C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$.*

ii) *If $\{\lambda_{n,p}\}$ denotes the set of eigenvalues to (2.2) obtained in Lemma 9 below then, for every n there exists a constant $B_n \in (0, 1]$ so that problem (1.2) admits at least n pairs $\pm u_{k,n}(\lambda)$, $1 \le k \le n$, of nontrivial solutions for every*

$$\lambda > \frac{n^p}{B_n}\lambda_{n,p}. \tag{1.7}$$

We come now to the sublinear case $q < p$. We recall that some aspects of nonnegative solutions (particularly, dead core formation) to the linear diffusion problem (1.1) with $1 < q < 2$ were studied in detail in [13].

Theorem 5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded and class $C^{1,\alpha}$ domain and*

$$1 < q < p.$$

Then, problem (1.2) satisfies the following properties.

i) *There exists at least a* nonnegative *solution $u_\lambda \in C^{1,\beta}(\overline{\Omega})$ for every $\lambda > 0$. Moreover, the family $\{u_\lambda\}$ satisfies*

$$\|u_\lambda\|_{L^q(\Omega)} = O(\lambda^{-\frac{1}{p-q}}), \tag{1.8}$$

as $\lambda \to \infty$.

ii) *If $\{u_\lambda\}$, $0 < \lambda < \bar{\lambda}$, is an* arbitrary *family of solutions (not necessarily nonnegative), then*

$$u_\lambda = (|\partial\Omega|/|\Omega|)^{-\frac{1}{p-q}}\,\lambda^{-\frac{1}{p-q}}\,(\pm 1 + o(1)), \tag{1.9}$$

in $C^{1,\beta}(\overline{\Omega})$ as $\lambda \to 0+$. In other words, $|u_\lambda|$ bifurcates from infinity at $\lambda = 0$.

Remark 4. The estimate (1.8) is considerably improved in the one dimensional case, in the sense that *all* possible families u_λ of nonnegative solutions satisfy

$$\|u_\lambda\|_\infty = O\left(\lambda^{-\frac{p'}{p-q}}\right), \tag{1.10}$$

as $\lambda \to \infty$, where $p' = p/(p-1)$. It is quite likely that the approach employed in [13] for achieving the validity of (1.10) on general domains Ω when $p = 2$, can be extended to cover the nonlinear diffusion case. This would show that all possible nonnegative solutions develop a "dead core" when λ is large. Nevertheless, we are not addressing here the analysis of dead cores.

We now state a version for the sublinear case of Theorems 2 and 4.

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{1,\alpha}$ bounded domain, and assume*

$$1 < q < p.$$

Then the following properties hold:

i) *If $p = 2$ and $\{\lambda_n\}$ stands for the set of Steklov eigenvalues, problem (1.1) admits at least n pairs $\pm u_{k,n}(\lambda)$, $1 \le k \le n$, of nontrivial solutions for every $\lambda > \lambda_n$.*

ii) *If $p \ne 2$ and $\{\lambda_{n,p}\}$ is the set of Steklov eigenvalues to (2.2) introduced in Lemma 9, problem (1.2) possesses at least n pairs $\pm u_{k,n}(\lambda)$, $1 \le k \le n$, of nontrivial solutions if*

$$\lambda > \frac{n^p}{B_n}\lambda_{n,p},$$

where the B_n's are the constants given in ii) of Theorem 4.

Remark 5. It should be observed that, in contrast to Theorem 2, we are not able to decide if any of the nontrivial solutions obtained in *i*) and *ii*) is two signed. In fact, it was shown in [13] that problem (1.1) in the sublinear case has a natural tendency to exhibit multiple non negative solutions as λ is large (even when Ω is a ball).

Remark 6. In the linear diffusion case $p = 2$, bifurcation from infinity at the Steklov eigenvalues $\lambda = \lambda_n$ seems to be the natural onset for the solutions introduced in point *i*). As stated below, such bifurcation occurs at all those λ_n whose multiplicity is odd. Again, if one considers domains Ω whose Steklov eigenvalues are all simple, then signed pairs of nontrivial solutions bifurcate from infinity as λ crosses the λ_n's. However, it is stressed that proving the continuation of those "local" branches throughout $\lambda > \lambda_n$ is by no means an obvious matter.

Theorem 7. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\alpha}$, $0 < \alpha < 1$, while $1 < q < 2$. Let $\bar\lambda$ be an arbitrary odd multiplicity eigenvalue to (2.1). Then, the following features hold.*

i) *For any $\lambda > \bar\lambda$, λ close to $\bar\lambda$, problem (1.1) admits a pair $(\lambda, \pm u_\lambda)$ of closed connected sets (branches) of nontrivial solutions such that*

$$\|u_\lambda\|_\infty \to \infty \quad \lambda \to \bar\lambda + .$$

ii) *Assume in addition that $\bar\lambda \ne 0$ is simple. Then, (1.1) admits a pair $(\lambda, \pm u_\lambda)$ of connected branches, bifurcated from infinity, such that*

$$u_\lambda = E^{-\frac{1}{2-q}}(\lambda - \bar\lambda)^{-\frac{1}{2-q}}(\bar\phi + o(1)) \quad \lambda \to \bar\lambda+, \tag{1.11}$$

in $C^2(\overline{\Omega})$, where $E = \int_{\partial\Omega}\bar\phi^2/\int_\Omega|\bar\phi|^q$ and $\bar\phi$ is an eigenfunction associated to $\bar\lambda$ so that $\|\bar\phi\|_\infty = 1$. In particular, bifurcated solutions change sign if $\bar\lambda \ne 0$.

Remark 7. In the case of odd eigenvalues $\bar{\lambda} \neq 0$ it can be shown that "most" of the bifurcated solutions are two signed. Specifically, every sequence of nontrivial solutions $(\lambda_k, u_k) \to (\bar{\lambda}, \infty)$ admits a subsequence, still labeled u_k, such that $u_k = E^{-\frac{1}{2-q}}(\lambda_k - \bar{\lambda})^{-\frac{1}{2-q}}(\bar{\phi} + o(1))$, with a certain eigenfunction $\bar{\phi}$ associated to $\bar{\lambda}$, $\|\bar{\phi}\|_\infty = 1$ and E as in (1.11).

The previous results can be extended to the more general class of spatially heterogeneous problems

$$\begin{cases} \Delta_p u = a(x)\varphi_q(u), & x \in \Omega, \\ \mathcal{B}_p(u) = \lambda \varphi_p(u), & x \in \partial\Omega, \end{cases} \tag{1.12}$$

where $a \in C(\overline{\Omega})$ is a nonnegative function which could even become identically zero on a whole (smooth) subdomain Ω_0 of Ω. In this last case, a suitable restriction must be imposed in the range of variation of the parameter λ. A detailed account on the extensions to problem (1.12) of Theorems 1–7 is included in Sect. 5.

The rest of the paper is organized as follows: Sect. 2 includes some basic material on critical point theory (Sect. 2.1), a description of the Steklov eigenvalue problem both in its linear and nonlinear diffusion versions (Sect. 2.2) and L^∞ estimates for the solutions to problems (1.1) and (1.2) (Sect. 2.3). The proofs of the statements corresponding to the superlinear case $q > p$ and the sublinear case are separately collected in Sects. 3 and 4, respectively. Finally, Sect. 5 is devoted to the analysis of problem (1.12).

2. Some definitions and auxiliary results

2.1. Critical points of functionals

Let us introduce some minimum background on index theory and critical points of C^1 functionals in Banach spaces required for later proofs (see Chapter II in [27] for an overview). In the present section X stands for an infinite dimensional (in most occasions) Banach space. Concerning index theory, the Krasnosel'skii genus γ on X ([3,23,27]) is a topological index $\gamma : \Sigma \to \mathbb{N} \cup \{0, \infty\}$ defined on the class Σ of the symmetric closed parts of X: $\Sigma = \{C \subset X : C = \overline{C},\ C = -C\}$, as follows: For $C \in \Sigma$, $\gamma(C) = k$ if k is the minimum integer l so that a continuous odd function $h : C \to \mathbb{R}^l \setminus \{0\}$ exists. We set $\gamma(\emptyset) = 0$ while, for obvious reasons, $\gamma(C) = \infty$ if $0 \in C$. To catch an insight on the genus it should be first mentioned that γ remains invariant under homeomorphisms and exhibits suitable subadditivity and stability properties (see [3,27]). On the other hand, dimension and genus are somehow related. If $\dim X = m$ and $A = -A$ is a neighborhood of 0 then $\gamma(\partial A) = m$ meanwhile, for X arbitrary, $\gamma(A) = m$ implies that A contains m linearly independent vectors [27]. In addition, we have that A is an infinite set provided $\gamma(A) \geq 2$, $0 \notin A$.

A result providing the existence of critical points for a class of C^1 functionals $J : X \to \mathbb{R}$ defined in a Banach space X is going to be stated in a moment. Conditions required to that class of functionals J are

J-i) J satisfies the Palais–Smale condition (PS for short), that is, from every sequence x_n with $J(x_n)$ bounded and $DJ(x_n) \to 0$ it is possible to extract a subsequence which converges in X,

J-ii) there exist α, ρ positive and $e \in X \setminus \{0\}$ such that $J > 0$ for $0 < |x| < \rho$ and $J(x) \geq \alpha$ if $|x| = \rho$, while $J(e) = 0$,

J-iii) $J(-x) = J(x)$ for all $x \in X$ (J is even),

J-iv) there exists an increasing sequence of subspaces $X_m \subset X$, dim $X = m$, and a compact set $K_m \subset X_m$ in each X_m so that 0 lies in a bounded component of $X_m \setminus K_m$ while $J(x) < 0$ for all $x \in K_m$.

Observe that $i)$, $ii)$ are conditions that provide the validity of the so-called mountain pass lemma ([3,24]).

We next set $\mathrm{Hom}_{\mathrm{odd}}$ the class of all odd homeomorphisms $h \in C(X, X)$, $h(-x) = -h(x)$ for all $x \in X$, together with

$$\Gamma = \{h \in \mathrm{Hom}_{\mathrm{odd}} : J \geq 0 \text{ in } h(B)\},$$

with B the unit ball of X. Under hypothesis J-$iv)$ it is shown in [3] (p. 361) that the class

$$\Gamma_m = \{K \subset X : K = -K, K \text{ compact}, \gamma(K \cap h(\partial B)) \geq m \text{ for all } h \in \Gamma\}$$

is nonempty for all the integers m involved in J-$iv)$.

Now we can introduce the desired result on existence of critical points of J (for the proof we refer to [3], Theorem 2.23).

Theorem 8. *Assume that J satisfies J-i) to J-iv). Then*

$$c_m := \inf_{K \in \Gamma_m} \max_{x \in K} J(x)$$

defines a critical value for every m (and $\alpha \leq c_m < \infty$ for every m). Moreover, if $c_m = \cdots = c_{m+l-1}$ for some $l \in \mathbb{N}$, then $\gamma(K_{c_m}) \geq l$ where $K_{c_m} := \{x : J(x) = c_m$ and $DJ(x) = 0\}$.

Remark 8. The last assertion in the theorem implies the existence of infinitely many critical points corresponding to those critical values c_m with multiplicity l higher than 2 ($l \geq 2$).

2.2. Eigenvalue problems

The Steklov eigenvalue problem

$$\begin{cases} \Delta u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

consists in finding real values λ, and corresponding non trivial functions $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla \psi = \lambda \int_{\partial\Omega} u\psi,$$

for all $\psi \in H^1(\Omega)$. As it is readily seen, Steklov eigenvalues are precisely the eigenvalues to the Dirichlet to Neumann operator in Ω. It is well-known that the totality of such eigenvalues consists of an increasing sequence

$$\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots,$$

$\lambda_n \to \infty$, each λ_k repeated according its (finite) multiplicity, such that an associated sequence of eigenfunctions $\phi_1, \phi_2, \ldots$ can be formed so that it provides a complete orthonormal system in $H^1(\Omega)$ (see, for instance, [8,18]). For subsequent use here, eigenfunctions ϕ_k are normalized so as

$$\int_{\partial\Omega} \phi_i \phi_j = \delta_{i,j} \quad i,j \in \mathbb{N},$$

while we are using $(u,v) = \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} uv$ as scalar product in $H^1(\Omega)$.

The "nonlinear diffusion" counterpart of the Steklov problem will be also involved in the present work. Namely, the problem

$$\begin{cases} \Delta_p u = 0, & x \in \Omega, \\ \mathcal{B}_p(u) = \lambda \varphi_p(u), & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where $\mathcal{B}_p$ stands for the boundary operator $\mathcal{B}_p(u) = |\nabla u|^{p-2} \frac{\partial u}{\partial \nu}$ while $\varphi_p(u) = |u|^{p-2}u$ $(p > 1)$. In a similar way as in the linear case, λ is called an eigenvalue to (2.2) provided a non trivial $u \in W^{1,p}(\Omega)$ exists such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \psi = \lambda \int_{\partial\Omega} \varphi_p(u)\psi,$$

for each $\psi \in W^{1,p}(\Omega)$. However, and in strong difference with (2.1) only partial features on the spectrum of (2.2) are currently available. It is known that $\hat{\lambda}_1 = 0$ is the minimum eigenvalue which is simple, isolated and the only one with a positive associated eigenfunction (which is indeed constant). On the other hand, further eigenvalues $\lambda = \hat{\lambda}_n$ can be found by employing the general procedure in [2] (see [4,11] for the case of Dirichlet eigenvalues, and [9] for a perturbation by a potential term of (2.2)). More precisely, for $n \in \mathbb{N}$ set

$$\frac{\alpha}{\hat{\lambda}_n + 1} := \beta_n, \tag{2.3}$$

where the value β_n is variationally obtained as

$$\beta_n = \inf_{C \in \mathcal{C}_n} \sup_{u \in C} b(u)$$

with $b(u) = \int_{\partial\Omega} |u|^p$, $\mathcal{M} = \{u \in W^{1,p}(\Omega) : \|u\|_{1,p}^p = \alpha\}$, $\alpha > 0$ is a certain constant, $\|u\|_{1,p}^p := \int_\Omega |\nabla u|^p + \int_{\partial\Omega} |u|^p$, and

$$\mathcal{C}_n = \{C \subset \mathcal{M} : C = -C,\ C \text{ compact},\ \gamma(C) \geq n\},$$

where $\gamma(C)$ stands for Krasnosel'skii genus of C (Sect. 2.1). In fact, it is shown in [2] that every β_n defines a critical value for the functional $b(u)$ on the manifold $\mathcal{M}$. Therefore, the corresponding critical point $u_n \in \mathcal{M}$ defines an associated eigenfunction corresponding to $\lambda = \hat{\lambda}_n$ given through (2.3). Moreover, it is also proved there that in case $\beta_n = \cdots = \beta_{n+(l-1)}$ with $l \geq 2$, then the critical level $\{b(u) = \beta_n\} \cap \mathcal{M}$ indeed contains infinitely many critical points.

On the other hand, proceeding as in [9] (see also [11]), it can be shown that $\beta_n \to 0$. This implies that (2.2) admits an infinite sequence $\hat{\lambda}_n \to \infty$ of eigenvalues. However, at the present moment it is a (hard) open problem to ascertain whether these $\hat{\lambda}_n$ fill up the whole set of eigenvalues to (2.2). It is even unknown if such spectrum is discrete or not. The only shown spectral gap lies between $\hat{\lambda}_1$ and $\hat{\lambda}_2$ (see [10]), i. e. $\hat{\lambda}_2$ is the first eigenvalue after $\hat{\lambda}_1 = 0$. The distribution of the remaining eigenvalues remains "terra incognita" (needless to say that the very same features were first encountered in the Dirichlet spectrum [21]).

For future reference we term $\{\hat{\lambda}_n\}$ the LS-spectrum of (2.2). At the best of our knowledge it is not even clear if the whole set $\{\phi_n\}$ of associated eigenfunctions is linearly independent in some sense. A much weaker result in this direction—still useful for our purposes—is next stated.

Lemma 9. *The LS set $\{\hat{\lambda}_n\}$ of eigenvalues to* (2.2) *possesses an infinite* maximal *subsequence $\{\lambda_{n,p}\}$ such that the corresponding (normalized) associated eigenfunctions $\{\phi_{n,p}\}$ are independent in $L^p(\partial\Omega)$.*

Proof. It is enough to prove the existence of an infinite set of independent eigenfunctions. The assertion then follows from routine arguments in algebra. Thus, suppose on the contrary that the whole set $\{u_n\}$ of eigenfunctions lie in a finite dimensional subspace Z of $L^p(\partial\Omega)$. Functions u_n can be normalized so that $(1+\hat{\lambda}_n)^{1/(p-1)}u_n$ satisfy $\|(1+\hat{\lambda}_n)^{1/(p-1)}u_n\|_{L^p(\partial\Omega)} = 1$. We now observe that weak and strong topologies coincide in Z. Thus, after passing to a subsequence, $(1+\hat{\lambda}_n)^{1/(p-1)}u_n \to v$ in $L^p(\partial\Omega)$.

Set now $B_p : W^{1-\frac{1}{p},p}(\partial\Omega)^* \to W^{1,p}(\Omega)$ ("$*$" meaning "dual space") the operator associating to $g \in W^{1-\frac{1}{p},p}(\partial\Omega)^*$ the unique solution u to the problem

$$\begin{cases} \Delta_p u = 0, & x \in \Omega, \\ \mathcal{B}_p(u) + \varphi_p(u) = g, & x \in \partial\Omega. \end{cases}$$

It can be shown by using the monotonicity properties of the operator $-\Delta_p$ (see [28]) that B_p is continuous. On the other hand

$$u_n = B_p\left(\varphi_p((1+\hat{\lambda}_n)^{1/(p-1)}u_n)\right).$$

Since $\varphi_p((1+\hat{\lambda}_n)^{1/(p-1)}u_n) \to \varphi_p(v)$ in $L^{p'}(\partial\Omega)$ then $u_n \to u$ in $W^{1,p}(\Omega)$ where u solves the previous problem with $g = \varphi_p(v)$ and so

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\psi + \int_{\partial\Omega} \varphi_p(u)\psi = \int_{\partial\Omega} \varphi_p(v)\psi, \quad \psi \in W^{1,p}(\Omega).$$

Observe in addition that $u_n = (1+\hat{\lambda}_n)^{-1/(p-1)}(1+\hat{\lambda}_n)^{1/(p-1)}u_n \to 0$ in $L^p(\partial\Omega)$ and so $u = 0$ on $\partial\Omega$. Thus, by setting $\psi = u$ we get $\int_\Omega |\nabla u|^p = 0$, i. e. $u = 0$ in the whole of Ω. But putting instead $\psi = v$ in the previous relation we then get $\|v\|^p_{L^p(\partial\Omega)} = 0$ which is not possible. This means that span $\{u_n\}$ can not be finite dimensional in $L^p(\partial\Omega)$. □

Remark 9. The set $\{\phi_{n,p}\}$ introduced in Lemma 9—and accordingly the associated eigenvalue set—can be chosen so that it contains a prefixed linearly independent set of eigenfunctions. It is well known that every eigenfunction

ϕ_2 associated to $\hat\lambda_2$ exhibits at least two nodal domains. Therefore, $\hat\lambda_1, \hat\lambda_2$ can be included in $\{\lambda_{n,p}\}$.

2.3. Some estimates

Next, we show that weak solutions to (1.2) are indeed essentially bounded in Ω. This allows us to introduce a truncation of (1.2) which is instrumental in the analysis of the superlinear case $q > p$.

A proof of the following result is included for its use in what follows.

Proposition 10. *Let $u \in W^{1,p}(\Omega)$ be a weak solution of (1.2). Then $u \in L^\infty(\Omega)$. Moreover, there exists a positive $M = M(p, \Omega, \lambda, \|u\|_{L^1(\Omega)})$ which does not depend on q so that*

$$\|u\|_{L^\infty(\Omega)} \le M.$$

Proof. We can assume that $p \le N$. To show that u^+ is bounded (the same reasoning applies to u^-) we first obtain an estimate of the form

$$\int_{A_k} (u-k) \le Ck|A_k|^{1+\frac{1}{N}}, \tag{2.4}$$

for $k \ge k_0 = k_0(\lambda, \|u\|_{L^1(\Omega)})$, where $A_k = \{u(x) > k\}$ and $C = C(\lambda)$. Then the key tool is Lemma 5.1 of Chapter II in [19] which states that (2.4) implies $u^+ \in L^\infty(\Omega)$. We briefly describe the reasoning in [19]. One first observes that

$$F(t) = \int_{A_t} (u-t) = \int_t^\infty |A_s|\, ds,$$

and so $F'(t) = -|A_t|$ a. e. in $t > k_0$. Thus, (2.4) can be rewritten as

$$C^{\frac{1}{\eta+1}} \frac{d}{dt}\left(F^{\frac{\eta}{\eta+1}} \right) \le -\frac{d}{dt}\left(t^{\frac{\eta}{\eta+1}} \right)$$

a.e. in $t \ge k_0$ with $\eta = \frac{1}{N}$. By integrating this relation between k_0 and t one concludes $F = 0$ for $t \ge t_1 \ge k_0$ for a certain t_1 only depending on k_0, i. e. on $\|u\|_{L^1(\Omega)}$. This implies that $u^+ \le t_1$ a. e. in Ω.

To show (2.4) we use $\psi = (u-k)^+$ as test function in (1.3), with $k > 0$, to obtain that

$$\int_{A_k} |\nabla u|^p \le C \int_{\partial\Omega} ((u-k)^+)^p + Ck^{p-1} \int_{\partial\Omega} (u-k)^+, \tag{2.5}$$

where $C = C(\lambda)$. This inequality explains why the term $\varphi_q(u)$ in (1.2) does not play any rôle in the final estimate.

We now refresh a well known inequality: for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that

$$\int_{\partial\Omega} |v|^p \le \varepsilon \int_\Omega |\nabla v|^p + C_\varepsilon \int_\Omega |v|^p, \tag{2.6}$$

for all $v \in W^{1,p}(\Omega)$. By using (2.6) with $v = (u-k)^+$ and a suitable $\varepsilon = \varepsilon(\lambda)$, (2.5) leads to

$$\|(u-k)^+\|_{1,p}^p \le C \int_\Omega (u-k)^{+p} + Ck^{p-1} \int_{\partial\Omega} (u-k)^+, \tag{2.7}$$

with $C = C(\lambda)$, where the alternative norm $\|v\|_{1,p}^p = \int_\Omega |\nabla v|^p + \int_{\partial\Omega} |v|^p$ is going to be used. We now observe that

$$\int_\Omega ((u-k)^+)^p \le C|A_k|^{1-\frac{p}{p^*}} \|(u-k)^+\|_{1,p}^p, \tag{2.8}$$

where we assume that $p < N$ (a similar idea works when $p = N$). Since

$$|A_k| \le \frac{1}{k}\|u\|_{L^1(\Omega)},$$

there exists $k_0 = k_0(C, \|u\|_{L^1(\Omega)})$ such that, for $k > k_0$, $C|A_k|^{1-\frac{1}{p^*}}$ becomes less than one in (2.8). This fact and (2.7) yield

$$\begin{aligned}\|(u-k)^+\|_{1,p}^p &\le Ck^{p-1}\int_{\partial\Omega}(u-k)^+ \le Ck^{p-1}\|(u-k)^+\|_{W^{1,1}(\Omega)}\\ &\le Ck^{p-1}|A_k|^{1-\frac{1}{p}}\|(u-k)^+\|_{1,p},\end{aligned}$$

which implies that $\|(u-k)^+\|_{1,p} \le Ck|A_k|^{\frac{1}{p}}$, for $k \ge k_0$. Thus,

$$\int_{A_k}(u-k) \le C|A_k|^{1-\frac{1}{p^*}}\|(u-k)^+\|_{1,p} \le Ck|A_k|^{1+\frac{1}{p}-\frac{1}{p^*}},$$

as desired. □

Now we prove that in the superlinear regime $q > p$, problem (1.2) admits a suitable truncation which possesses the same solutions as (1.2). For $m > 0$ we define $h_m(u)$ as the odd function such that

$$h_m(u) = \begin{cases} u^{q-1}, & 0 \le u \le m,\\ \frac{q-1}{p-1}m^{q-p}u^{p-1} - \frac{q-p}{p-1}m^{q-1}, & u > m.\end{cases}$$

and we also set $H_m(u) = \int_0^u h_m$. Observe that $pH_m(u) - h_m(u)u \le 0$ in $\mathbb{R}$. Let us introduce the truncated problem

$$\begin{cases} \Delta_p u = h_m(u), & x \in \Omega,\\ \mathcal{B}_p(u) = \lambda\varphi_p(u), & x \in \partial\Omega,\end{cases} \tag{2.9}$$

Proposition 11. *Assume Ω satisfies the conditions of Theorem 1, $q > p$ while λ is fixed. Then there exists $m_0 = m_0(\lambda) > 0$ such that problems (1.2) and (2.9) have exactly the same solutions for $m \ge m_0$.*

Proof. Let us introduce

$$\widetilde{J}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_{\partial\Omega}|u|^p + \int_\Omega H_m(u),$$

$u \in W^{1,p}(\Omega)$, the energy functional associated to (2.9). We first show that all possible solutions to (2.9) lie in the level set $\{\widetilde{J}(u) \le 0\}$. In fact,

$$\widetilde{J}(u) = \frac{1}{p}\langle D\widetilde{J}(u), u\rangle + \int_\Omega\left(H_m(u) - \frac{1}{p}h_m(u)u\right) = \int_\Omega\left(H_m(u) - \frac{1}{p}h_m(u)u\right) \le 0$$

since $pH_m(u) - h_m(u)u \le 0$ for all u.

Let us analyze now the coercivity properties of $\widetilde{J}$. A proper use of (2.6) allows us to write

$$\widetilde{J}(u) \geq \frac{1}{2p}\int_\Omega |\nabla u|^p + \int_\Omega H_m(u) - C_1 \int_\Omega |u|^p,$$

for a certain constant C_1. On the other hand, there exists a constant $C_2 > 0$ such that

$$\frac{u^q}{q} > 2C_1 u^p - C_2,$$

for all $u > 0$. Then, the existence of a value $m_0 = m_0(p,q)$ (as large as desired) can be shown so that

$$H_m(u) > 2C_1 u^p - C_2,$$

for all $u > 0$ provided $m \geq m_0$. Thus,

$$\widetilde{J}(u) \geq C\|u\|_{1,p}^p - C_3, \quad u \in W^{1,p}(\Omega). \tag{2.10}$$

Hence, recalling that solutions u satisfy $\widetilde{J}(u) \leq 0$, there exists a constant K_1 which does not depend on m such that all possible solutions to (2.9) with $m \geq m_0$ verify the uniform estimate $\|u\|_{1,p} \leq K_1$. We notice now that Proposition 10 and its proof can be applied step by step with no changes ($h(u)$ has no influence in the computations) to conclude the existence of a constant K_2 not depending on m so that all solutions to (2.9) satisfy $\|u\|_\infty \leq K_2$. Therefore and by enlarging m_0 if necessary, solutions u to (2.9) also solve (1.2) for all $m \geq m_0$.

Conversely, the same argument as the previous one proves that all possible solutions to (1.2) lie on the level set $J(u) \leq 0$ (recall that solutions belong to $L^q(\Omega)$ so that $J(u)$ is well-defined). This fact combined with the coercivity of J (the functional J also satisfies (2.10)) implies that solutions of (1.2) are uniformly bounded in Ω and so give solutions to (2.9) for large m.– □

Remarks 10. (a) A careful examination of the proof above shows that if $0 < \lambda \leq \Lambda$, the value m_0 can be chosen depending only on Λ.

(b) For later use, we observe that thanks to the last part of the proof of Proposition 11 and estimates in [20], solutions u to (1.2) lie in $C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$ and are uniformly bounded in that space, i.e., there exists $M = M(\lambda) > 0$ so that every weak solution u to (1.2) satisfies

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq M. \tag{2.11}$$

For any $\Lambda > 0$ fixed and in view of (a), M can be chosen no depending on λ for $0 < \lambda \leq \Lambda$.

3. The superlinear case $q > p$

Proof of Theorem 1. Since weak solutions to (1.2) must satisfy (just take $\psi = u$ in (1.3))

$$\int_\Omega |\nabla u|^p + |u|^q = \lambda \int_{\partial\Omega} |u|^p,$$

non trivial solutions only occur when $\lambda > 0$. In addition, existence of a non-negative solution to (1.2) follows by the argument in the proof of Theorem 2 in [12]. In fact, we deal directly with the truncated problem (2.9) with $m \geq m_0$ (Proposition 11) and observe that $\widetilde{J}$ has a non trivial absolute minimizer u in $W^{1,p}(\Omega)$ ($\widetilde{J}$ is coercive, weakly semicontinuous, bounded below and, moreover, $\widetilde{J}(c) < 0$ for a small constant c). Since $H_m(|u|) = H_m(u)$, u can be chosen non-negative. Moreover, $u \in C^{1,\beta}(\overline{\Omega})$ (Remark 10 (b)) and the strong maximum principle then gives that $u > 0$ in $\overline{\Omega}$.

Regarding the uniqueness of nonnegative solutions we delay its proof until Sect. 5 (proof of Theorem 12) where we deal with a more general case. We remark that we proceed in the proof of Theorem 12 by an alternative approach to the one in Lemma 8 in [12] for the case $p = 2$.

Set u_λ the positive solution to (1.2). Now one observes that if $u \geq 0$ solves (1.2) then Mu becomes either a subsolution or a supersolution provided $M \leq 1$ or $M \geq 1$, respectively (see [15] for a version of the method of sub and supersolutions adapted to problems with nonlinear boundary conditions as (1.2)). Since u_{λ_1} becomes a supersolution to (1.2) if $\lambda_1 > \lambda$ then it follows that $u_\lambda \leq u_{\lambda_1}$ and the monotonicity in λ is proved. Continuous dependence on λ (with values in $C^{1,\beta}(\overline{\Omega})$) is now a consequence of the uniqueness and the estimates in [20] (see Sect. 2.3, specially Remarks 10 (b)).

To complete *ii*) we take $\lambda_n \in (0, \bar{\lambda})$, $\lambda_n \to 0$ and set $u_n = u_{\lambda_n}$. Since $0 < u_n < u_{\bar{\lambda}}$, u_n is uniformly bounded, so by the estimates in [20] and modulus a subsequence we get $u_n \to u$ in $C^{1,\beta}(\overline{\Omega})$. The limit u solves (1.2) with $\lambda = 0$. Thus, $u = 0$ and furthermore $u_\lambda \to 0$ as $\lambda \to 0$. To refine this behavior we now take (as usual) $t_n = \|u_n\|_\infty$, $u_n = t_n v_n$ and by substituting in (1.2) and taking limits we get $v_n \to 1$ in $C^{1,\beta}(\overline{\Omega})$. In other words,

$$u_n = t_n(1 + o(1))$$

in $C^{1,\beta}(\overline{\Omega})$. In addition, by inserting $\psi = 1$ in (1.3) one finds

$$\lambda_n t_n^{p-q} \int_{\partial\Omega} \varphi_p(v_n) = \int_\Omega \varphi_q(v_n).$$

This implies that

$$t_n = E^{\frac{1}{q-p}} \lambda_n^{\frac{1}{q-p}} (1 + o(1)),$$

with $E = |\partial\Omega|/|\Omega|$. This combined with the previous expression for u_n gives the desired expression for u_λ.

A proof of point *iii*) on the behavior of u_λ as $\lambda \to \infty$ together with the corresponding result of existence of a minimal solution to the singular problem (1.5) is contained in [14]. □

Proof of Theorem 2. We begin with the analysis of the nonlinear diffusion case. In view of Proposition 11 we work directly with the truncated version (2.9) of problem (1.2) and show that $\widetilde{J}$ satisfies the PS condition. Accordingly if $u_n \in W^{1,p}(\Omega)$ satisfies $\widetilde{J}(u_n) \leq c$, then, coercivity, see (2.10), of $\widetilde{J}$ implies that u_n is bounded in $W^{1,p}(\Omega)$ and, modulus a subsequence, $u_n \to u$ weakly in $W^{1,p}(\Omega)$. Setting $v_n = u_n - u$ we observe (as usual) that

$$\langle -\Delta_p u_n, v_n\rangle = \langle D\widetilde{J}(u_n), v_n\rangle + \lambda \int_{\partial\Omega} \varphi_p(u_n)v_n - \int_\Omega h(u_n)v_n = o(1),$$

provided that u_n satisfies in addition $D\widetilde{J}(u_n) \to 0$, which is the complementary condition in order that u_n is a PS sequence. Thus,

$$\langle -\Delta_p u_n - (-\Delta_p u), u_n - u\rangle \to 0,$$

and the monotonicity of the p-Laplacian implies that $u_n \to u$ in $W^{1,p}(\Omega)$.

Let us come back to the linear case $p = 2$ and study the geometry of $\widetilde{J}$ near zero. To this end set $X_n = \text{span}\{\phi_1, \dots, \phi_n\}$, where $\phi_1, \dots, \phi_n$ are the first n-Steklov eigenfunctions, normalized as in Sect. 2.2. We now observe that, for all $u = \sum_{i=1}^n t_i\phi_i \in X_n$,

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega} |u|^2 = \frac{1}{2}\sum_{i=1}^n (\lambda_i - \lambda)t_i^2.$$

Hence, if $\lambda > \lambda_n$ and assuming that $|\bar{t}|_2^2 := \sum_{i=1}^n t_i^2$ is small, we find that for $u = \sum_{i=1}^n t_i\phi_i \in X_n$

$$\begin{aligned}\widetilde{J}(u) &= \frac{1}{2}\sum_{i=1}^n (\lambda_i - \lambda)t_i^2 + \frac{1}{q}|\bar{t}|_2^q \int_\Omega |\sum_{i=1}^n \frac{t_i}{|\bar{t}|_2}\phi_i|^q \\ &\le \frac{1}{2}(\lambda_n - \lambda)|\bar{t}|_2^2 + C|\bar{t}|_2^q < 0, \qquad (3.12)\end{aligned}$$

for a certain positive C, all $\bar{t} \in \mathbb{R}^n$ with $|\bar{t}|_2 = R$ and R sufficiently small.

For $X = W^{1,p}(\Omega)$ we introduce the class

$$\mathcal{C}_m = \{K \subset X : K = -K,\ K \text{ compact},\ \gamma(K) \ge m\},$$

where $\gamma(K)$ is the Krasnosel'skii genus of K (see Sect. 2.1). The sphere $K_{n,R} := \{u \in X_n : |\bar{t}|_2 = R\}$ is compact and $K_{n,R} \in \mathcal{C}_n$ since $\gamma(K_{n,R}) = n$ (we refer again to Sect. 2.1). In view of the previous analysis we have $\max_{K_{n,R}} J < 0$ for R small enough. Thus

$$-\infty < \tilde{c}_1 := \inf_X \widetilde{J} \le \tilde{c}_n := \inf_{K\in\mathcal{C}_n} \max_{u\in K} \widetilde{J}(u) < 0,$$

where the finiteness of $\tilde{c}_1$ is furnished by the coercive character of $\widetilde{J}$. Therefore, Theorem II.5.7 in [27] states that $\tilde{c}_n$ is a critical value of $\widetilde{J}$. Moreover, in case of multiplicity l higher than 2, that is, if $\tilde{c}_n = \cdots = \tilde{c}_{n+l-1}$ for a certain $l \in \mathbb{N}$, $l \ge 2$, then $\gamma(K_{\tilde{c}_n}) \ge l$ where $K_{\tilde{c}_n} = \{u \in X : \widetilde{J}(u) = \tilde{c}_n \text{ and } D\widetilde{J}(u) = 0\}$ ([27], Lemma II.5.6). In all cases, we conclude the existence of at least n distinct pairs $\pm u_{n,\lambda}$ of nontrivial solutions to (2.9) for all $\lambda > \lambda_n$. □

Proof of Theorem 4. That the whole set of solutions to (1.2) for $\lambda > 0$ fixed constitutes a compact set in $C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$ follows from Proposition 11 (see Remarks 10 (b)).

On the other hand, the proof of *ii*) is similar to the previous one. We set instead, $X_{n,p} = \text{span}\{\phi_{1,p}, \dots, \phi_{n,p}\}$, the family of the n first LS eigenfunctions associated to the eigenvalues $\lambda_{1,p},\dots,\lambda_{n,p}$ introduced in Lemma 9. If $u = \sum_{i=1}^n t_i\phi_{i,p}$ is arbitrary in $X_{n,p}$, $\bar t = (t_i) \in \mathbb{R}^n$, then

$$\int_\Omega |\nabla u|^p - \lambda\int_{\partial\Omega}|u|^p = \int_\Omega |\sum_{i=1}^n t_i\nabla\phi_{i,p}|^p - \lambda\int_{\partial\Omega}|\sum_{i=1}^n t_i\phi_{i,p}|^p$$
$$\le n^{p-1}\sum_{i=1}^n \lambda_{i,p}|t_i|^p - \lambda B_n|\bar t|_2^p \le \left(\frac{n^p}{B_n}\lambda_{n,p}-\lambda\right)|\bar t|_2^p, \tag{3.13}$$

where $B_n = \inf_{|\bar t|_2=1}\int_{\partial\Omega}|\sum_{i=1}^n t_i\phi_{i,p}|^p \le 1$. Notice that independence of the $\phi_{i,p}$'s is involved here in order to assert that $B_n > 0$. Thus, for all $u \in X_{n,p}$

$$\widetilde J(u) \le \frac{1}{p}\left(\frac{n^p}{B_n}\lambda_{n,p}-\lambda\right)|\bar t|_2^p + C|\bar t|_2^q < 0,$$

if $\bar t$ belongs to the the sphere $|\bar t|_2 = R$ in $X_{n,p}$, $R > 0$ is small enough and, of course, λ satisfies (1.7); that is,

$$\lambda > \frac{n^p}{B_n}\lambda_{n,p}.$$

The remaining part of the proof coincides with that of Theorem 2 (observe that it was shown there that $\widetilde J$ fulfills PS). In particular we also have (in view of Theorem 1) that $n-1$ of the obtained nontrivial pairs $\pm u_{k,n}(\lambda)$, say $2 \le k \le n$, correspond to two-signed solutions. □

Proof of Theorem 3. Let $\bar\lambda$ be an arbitrary Steklov eigenvalue with an associated system $\{\bar\phi_1,\dots,\bar\phi_n\}$ of eigenfunctions, normalized such that

$$(\bar\lambda+1)\int_{\partial\Omega}\bar\phi_i\bar\phi_j = \delta_{ij},\quad i,j\in\{1,\dots,n\},$$

and so they are orthonormal with respect to the scalar product

$$(u,v) = \int_\Omega \nabla u\nabla v + \int_{\partial\Omega} uv.$$

We use ideas from [25] (see also Chapter XI in [24]) to show that finding solutions (λ,u) to (1.1) with (λ,u) close to $(\bar\lambda,0)$ in $\mathbb{R}\times H^1(\Omega)$ amounts to searching for small amplitude critical points $v\in Y$, $Y := \operatorname{span}\{\bar\phi_1,\dots,\bar\phi_n\}$, of a finite dimensional real function $g(\lambda,v)$ defined in a neighborhood of $(\bar\lambda,0)$ in $\mathbb{R}\times Y$. This implies a reduction in the problem dimension and, of course, involves a Lyapunov–Schmidt reduction.

For this sake, it is convenient to work with the truncated problem (2.9) and the corresponding functional $\widetilde J$ (cf. Sect. 2.3). We set $R : H^1(\Omega)^* \to H^1(\Omega)$ the Riesz mapping associating to every $f\in H^1(\Omega)^*$ a unique $Rf\in H^1(\Omega)$ so that

$$\langle f,u\rangle = (Rf,u) \qquad u\in H^1(\Omega)$$

(here $(\cdot,\cdot)$ stands for the scalar product in $H^1(\Omega)$ and $\langle\cdot,\cdot\rangle$ for the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$). In addition, for $f\in H^1(\Omega)^*$, $g\in H^{1/2}(\partial\Omega)^*$

we consider the boundary value problems

$$\begin{cases} -\Delta u_1 = f, & x \in \Omega, \\ \dfrac{\partial u_1}{\partial \nu} + u_1 = 0, & x \in \partial\Omega, \end{cases} \qquad \begin{cases} \Delta u_2 = 0, & x \in \Omega, \\ \dfrac{\partial u_2}{\partial \nu} + u_2 = g, & x \in \partial\Omega, \end{cases}$$

whose respective solutions $u_1 := A(f)$, $u_2 := B(g)$ define continuous linear operators $A : H^1(\Omega)^* \to H^1(\Omega)$ and $B : H^{1/2}(\partial\Omega)^* \to H^1(\Omega)$, respectively. Thus, the "action" of $D\widetilde{J}$ can be written as

$$\langle D\widetilde{J}(u), \zeta\rangle = (u - (\lambda+1)B(u) + A(h_m(u)), \zeta) \quad \zeta \in H^1(\Omega).$$

Hence,

$$RD\widetilde{J}(u) = u - (\lambda+1)B(u) + A(h_m(u)),$$

and solving (1.1) is equivalent to solve

$$u - (\lambda+1)B(u) + A(h_m(u)) = 0, \tag{3.14}$$

in $H^1(\Omega)$. Notice that in this equation $B(u)$ stands for the composition of B with the trace mapping.

We set $\pi : H^1(\Omega) \to Y$ the orthogonal projection and for every $u \in H^1(\Omega)$ we set $u = \pi u + (I - \pi)u := v + w$. Then (3.14) is equivalent to

$$\begin{cases} \pi RD\widetilde{J}(v+w) = 0, \\ (I-\pi)RD\widetilde{J}(v+w) = 0. \end{cases} \tag{3.15}$$

The first equation in (3.15) can be written as

$$\frac{\bar{\lambda} - \lambda}{\bar{\lambda} + 1} v + \pi A(h_m(v+w)) = 0. \tag{3.16}$$

To get this expression, it has been used that for $u = v + w$, $B(v) = (\bar{\lambda}+1)^{-1}v$ while $B(w) \in Y^\perp$.

The second equation in (3.15) reads as

$$\mathcal{F}(\lambda, v, w) := w - (\lambda+1)B(w) + (I-\pi)A(h_m(v+w)) = 0, \tag{3.17}$$

where $\mathcal{F} : \mathbb{R} \times Y \times Y^\perp \to Y^\perp$ is a C^1 mapping with $\mathcal{F}(\bar{\lambda}, 0, 0) = 0$. In addition, the Fréchet derivative of $\mathcal{F}$ with respect to w at $(\bar{\lambda}, 0, 0)$ can be written as $D_w\mathcal{F}(\bar{\lambda}, 0, 0) = I_{Y^\perp} - (\bar{\lambda}+1)B$, where $I_{Y^\perp}$ is the restriction of the identity to $Y^\perp$. Due to the compactness of the trace mapping, the operator B is compact in $Y^\perp$, so that $D_w\mathcal{F}(\bar{\lambda}, 0, 0)$ is a one-to-one compact perturbation of the identity and hence, an isomorphism. The implicit function theorem then implies that the solutions (λ, v, w) to (3.17) which are close to $(\bar{\lambda}, 0, 0)$ have the form $w = \psi(\lambda, v)$ where $\psi : (\bar{\lambda} - \varepsilon_0, \bar{\lambda} + \varepsilon_0) \times U \subset \mathbb{R} \times Y \to Y^\perp$, $(\bar{\lambda} - \varepsilon_0, \bar{\lambda} + \varepsilon_0) \times U$ can be taken to be a symmetric open neighborhood of $(\lambda, v) = (\bar{\lambda}, 0)$ in $\mathbb{R} \times Y$, and ψ is a class C^1 mapping such that

$$\psi(\lambda, 0) = 0, \quad D_v\psi(\lambda, 0) = 0, \quad \psi(\lambda, -v) = -\psi(\lambda, v)$$

for every $\lambda \in (\bar{\lambda}-\varepsilon_0, \bar{\lambda}+\varepsilon_0)$, $v \in U$. Therefore, solutions (λ, u) to (3.14) close to $(\bar{\lambda}, 0)$ are $u = v + \psi(\lambda, v)$ with v solving

$$\frac{\bar{\lambda}-\lambda}{\bar{\lambda}+1} v + \pi A(h_m(v+\psi(\lambda, v))) = 0. \tag{3.18}$$

We now define $g : (\bar{\lambda}-\varepsilon_0, \bar{\lambda}+\varepsilon_0) \times U \to \mathbb{R}$ as $g(\lambda, v) = \widetilde{J}(v+\psi(\lambda, v))$ and prove that (3.18) furnishes its critical points. In fact, for every $\xi \in Y$:

$$\begin{aligned}\langle D_v g(\lambda, v), \xi\rangle &= \langle D\widetilde{J}(v+\psi), \xi + D_v\psi\xi\rangle = \left(R\, D\widetilde{J}(v+\psi), \xi + D_v\psi\xi\right) \\ &= \left(\pi R\, D\widetilde{J}(v+\psi), \xi\right) = \left(\frac{\bar{\lambda}-\lambda}{\bar{\lambda}+1} v + \pi A(h_m(v+\psi(\lambda, v))), \xi\right),\end{aligned}$$

where it has been used that $(I-\pi)R\,DJ(v+\psi) = 0$. Thus,

$$D_v g(\lambda, v) = \frac{\bar{\lambda}-\lambda}{\bar{\lambda}+1} v + \pi A(h_m(v+\psi(\lambda, v))),$$

and (3.18) gives the critical points of g.

We now claim that $v = 0$ is a strict local minimum and also an isolated critical point of $g(\bar{\lambda}, \cdot)$. If the claim is assumed then Theorem 2.2 in [25] implies the existence of $\delta_0 > 0$ such that for every $\bar{\lambda} < \lambda < \bar{\lambda}+\delta_0$, $g(\lambda, \cdot)$ admits n pairs $\pm v_k(\lambda)$, $1 \le k \le n$, of nontrivial critical points (observe that g is even in v), such that $v_k(\lambda) \to 0$ as $\lambda \to \bar{\lambda}+$. Hence, *ii*) follows from the claim together with the fact that bifurcation from $u = 0$ at $\lambda = \bar{\lambda}$ takes place in $C^{1,\beta_1}(\overline{\Omega})$ for some $1 < \beta_1 < 0$ (see below for the proof of this assertion).

Let us prove the claim. First observe that, since critical points of g provide with solutions of (2.9), we can apply Proposition 11 to obtain that for large enough m, solutions of (2.9) with $|\lambda - \bar{\lambda}| < \varepsilon_0$ are actually solutions to (1.2) with $|\lambda - \bar{\lambda}| < \varepsilon_0$ (see Remarks 10 (a)). Thus we may replace h_m by φ_q in what follows and so

$$g(\lambda, v) = \frac{1}{2}\frac{\bar{\lambda}-\lambda}{\bar{\lambda}+1}\|v\|^2 + \frac{1}{2}\|\psi\|^2 - \frac{\lambda+1}{2}\int_{\partial\Omega}\psi^2 + \frac{1}{q}\int_\Omega |v+\psi|^q,$$

with $\|u\|^2 = (u, u)$. In fact, $(v, \psi) = 0$ which together with the fact that $v \in Y$ entails that

$$\int_{\partial\Omega} v\psi = 0.$$

Notice in addition that $g(\lambda, 0) = 0$. By setting $\lambda = \bar{\lambda}$, we get

$$\begin{aligned} g(\bar{\lambda}, v) &= \frac{1}{2}\int_\Omega |\nabla\psi|^2 - \frac{\bar{\lambda}}{2}\int_{\partial\Omega}\psi^2 + \frac{1}{q}\int_\Omega |v+\psi|^q \\ &\ge \frac{\bar{\lambda}'-\bar{\lambda}}{2}\int_{\partial\Omega}\psi^2 + \frac{1}{q}\int_\Omega |v+\psi|^q > 0,\end{aligned}$$

where $\bar{\lambda}'$ stands for the first eigenvalue greater than $\bar{\lambda}$. Thus, $v = 0$ is a strict local minimum of $g(\bar{\lambda}, \cdot)$. Regarding the isolation of $v = 0$ as a critical point of

$g(\bar{\lambda}, \cdot)$ assume, on the contrary, that there exists a sequence $v_m \in Y$ of critical points with $v_m \to 0$. By writing

$$v_m = \sum_{i=1}^{n} t_i^{(m)} \bar{\phi}_i, \quad \bar{t}_m = \left(t_i^{(m)}\right)_{1 \le i \le n}, \quad \bar{\tau}_m = \frac{\bar{t}_m}{|\bar{t}_m|_2} = \frac{\bar{t}_m}{\|v_m\|},$$

we get that, modulus a subsequence, $\bar{\tau}_m \to \bar{\tau} = (\tau_i)_{1\le i\le n} \in S_{n-1}$, S_{n-1} the Euclidean sphere in $\mathbb{R}^n$. On the other hand, since v_m solves (3.18) with $\lambda = \bar{\lambda}$ then

$$\left(v_m, \pi A(\varphi_q(v_m + \psi(\bar{\lambda}, v_m)))\right) = 0. \tag{3.19}$$

Taking $u_m = A(\varphi_q(v_m + \psi(\bar{\lambda}, v_m)))$ we obtain

$$(v_m, \pi u_m) = (v_m, u_m) = (v_m, A(\varphi_q(v_m + \psi(\bar{\lambda}, v_m)))) = \int_\Omega \varphi_q(v_m + \psi(\bar{\lambda}, v_m)) v_m.$$

From (3.19) it follows that

$$\int_\Omega \varphi_q \left(\frac{v_m}{\|v_m\|} + o(1)\right) \frac{v_m}{\|v_m\|} = 0,$$

for all m and taking limits we conclude that $\int_\Omega |v|^q = 0$ with

$$v = \sum_{i=1}^{n} \tau_i \bar{\phi}_i \neq 0,$$

which is impossible. Thus, $v = 0$ is an isolated critical point of $g(\bar{\lambda}, \cdot)$ and the proof of the claim is finished.

To complete the proof of $i)$ we notice that it follows from (3.18) that

$$\lambda - \bar{\lambda} = (\bar{\lambda} + 1)\|v\|^{q-2} \int_\Omega \varphi_q \left(\frac{v}{\|v\|} + o(1)\right) \frac{v}{\|v\|}$$

for all possible solutions (λ, v) near $(\bar{\lambda}, 0)$. By arguing as in the last part one concludes that the integral must be positive for all $\|v\|$ small enough. This shows that bifurcated solutions only occur on the right of $\bar{\lambda}$.

Let us show now that bifurcation from $(\bar{\lambda}, 0)$ actually occurs in the topology of $\mathbb{R} \times C^{1,\beta}(\overline{\Omega})$. In fact, if $(\lambda_k, u_k) \to (\bar{\lambda}, 0)$ in $\mathbb{R} \times H^1(\Omega)$, there exist $0 < \beta < 1$ and $M > 0$ so that the estimate

$$\|u_k\|_{C^{1,\beta}(\overline{\Omega})} \le M,$$

holds true (Remarks 10 (a) and (b)). Thus, for $0 < \beta_1 < \beta$ a subsequence $u_{k'}$ can be extracted so that $u_{k'} \to 0$ in $C^{1,\beta_1}(\overline{\Omega})$. This conclusion can be extended to the whole sequence u_k since $u_k \to 0$ in $H^1(\Omega)$, and so the assertion is shown.

For the proof of $iii)$ we assume that $\bar{\lambda}$ is simple. In this special case $Y = \text{span}\ \{\bar{\phi}\}$ and bifurcated solutions are $u = s\bar{\phi} + \psi(\lambda, s)$, with ψ a C^1 mapping which is orthogonal to $\bar{\phi}$, $\psi(\lambda, -s) = -\psi(\lambda, s)$, $\psi(\lambda, 0) = 0$ and $D_s\psi(\lambda, 0) = 0$. Thus, we can write $\psi(\lambda, s) = s\psi_1(\lambda, s)$ with an also C^1 mapping ψ_1. Performing similar computations as before, Eq. (3.18) becomes

$$\frac{\lambda - \bar{\lambda}}{\bar{\lambda} + 1} - |s|^{q-2} \int_\Omega \varphi_q(\bar{\phi} + \psi_1)\bar{\phi} = 0. \tag{3.20}$$

The implicit function theorem can be used again to solve the equation in the form $\lambda = \lambda(s)$ where λ is an even function $\lambda \in C(\bar{\lambda} - \delta_0, \bar{\lambda} - \delta_0)$ while $\lambda \in C^1(\bar{\lambda} - \delta_0, \bar{\lambda} - \delta_0) \setminus \{0\}$, $\delta_0 > 0$ small, and so that $\lambda(s)$ is increasing for $s > 0$. In addition, it follows from (3.20) that

$$\lambda(s) = \bar{\lambda} + (\bar{\lambda} + 1)|s|^{q-2} \left(\int_\Omega |\bar{\phi}|^q \, dx + o(1) \right), \tag{3.21}$$

as $s \to 0$. On the other hand, since $(\bar{\phi}, \bar{\phi}) = 1$ we have

$$\bar{\lambda} + 1 = \frac{1}{\int_{\partial\Omega} \bar{\phi}^2}.$$

Setting $E = \int_{\partial\Omega} \bar{\phi}^2 / \int_\Omega |\bar{\phi}|^q$ one obtains from (3.21)

$$s = \pm E^{\frac{1}{q-2}} (\lambda - \bar{\lambda})^{\frac{1}{q-2}} (1 + o(1)),$$

as $\lambda \to \bar{\lambda}$. By employing this expression in the representation $u = s(\bar{\phi} + \psi_1)$ of the small amplitude solutions we get the expression announced in *iii*). This finishes the proof. □

4. The sublinear case $1 < q < p$.

Proof of Theorem 5. We first show that the functional J also satisfies PS in the sublinear case $1 < q < p$. To this purpose it is enough to show that u_n becomes bounded when both $J(u_n)$ is bounded and $DJ(u_n) \to 0$, for an arbitrary sequence $u_n \in W^{1,p}(\Omega)$, since the remaining part of the proof coincides with the one in Theorem 2. Thus, under the previous assumptions on u_n, we have

$$\frac{1}{p} \left(\int_\Omega |\nabla u_n|^p - \lambda \int_{\partial\Omega} |u_n|^p \right) + \frac{1}{q} \int_\Omega |u_n|^q = O(1).$$

If u_n is not bounded then, necessarily, $t_n^p := \int_{\partial\Omega} |u_n|^p \to \infty$ and $v_n := t_n^{-1} u_n$ satisfies

$$\frac{1}{p} \left(\int_\Omega |\nabla v_n|^p - \lambda \right) + \frac{1}{q t_n^{p-q}} \int_\Omega |v_n|^q = O(t_n^{-p}),$$

what implies that v_n is bounded. From $DJ(u_n) \to 0$ it also follows that

$$\left(\int_\Omega |\nabla v_n|^p - \lambda \right) + \frac{1}{t_n^{p-q}} \int_\Omega |v_n|^q = o(t_n^{-(p-1)}),$$

and so

$$\int_\Omega |v_n|^q = o(t_n^{-(q-1)}),$$

so that $v_n \to 0$ in $L^q(\Omega)$. However, since $\{v_n\}$ is bounded in $H^1(\Omega)$, modulus a subsequence we have $v_n \to v$ in $L^p(\Omega)$ and in $L^p(\partial\Omega)$ with $\int_{\partial\Omega} |v|^p = 1$. This is not possible and so J fulfills the PS condition.

We now study the geometry of J near zero. By a suitable choice of $\varepsilon > 0$ in inequality (2.6) we obtain

$$J(u) \geq C_1 \left(\int_\Omega |\nabla u|^p + \int_{\partial\Omega} |u|^p \right) - C_2 \int_\Omega |u|^p + \frac{1}{q} \int_\Omega |u|^q \\ \geq C_1 \|u\|_{1,p} - C_2 \|u\|^p_{L^p(\Omega)} + C_3 \|u\|^q_{L^p(\Omega)} \geq C_1 \|u\|^p_{1,p} + C \|u\|^q_{L^p(\Omega)}, \tag{4.22}$$

for $\|u\|_{1,p} \leq \varepsilon_0$ and certain positive constants C, C_1, ε_0, with $\varepsilon_0 = \varepsilon_0(\lambda)$ small and C_1 independent of λ. Thus, J has a strict local minimum at $u = 0$ and $J(u) > C_1 \varepsilon_0^p$ for $\|u\|_{1,p} = \varepsilon_0$. In addition, $J(e_\lambda) = 0$ for $e_\lambda = t$ where $t = t(\lambda)$ is a suitable positive constant. Therefore, J satisfies J-i), J-ii) in Sect. 2.1. Then the mountain pass lemma [3] gives the existence of a nontrivial solution $\tilde{u}_\lambda$ to (1.2) for all $\lambda > 0$ which satisfies $J(\tilde{u}_\lambda) \geq C_1 \varepsilon_0^p$.

However, it can not be shown that the solution $\tilde{u}_\lambda$ so obtained is nonnegative, which is our main concern now. If instead of J one considers the functional

$$J^+(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{\lambda}{p} \int_{\partial\Omega} (u^+)^p + \frac{1}{q} \int_\Omega |u|^q, \tag{4.23}$$

it is shown by the previous argument that J^+ fulfills PS and, since $J^+(u) \geq J(u)$, that J^+ has also the mountain pass lemma geometry near zero. Then J^+ admits a non trivial critical point u_λ satisfying $J^+(\tilde{u}_\lambda) \geq C_1 \varepsilon_0^p$. Moreover, by setting $\psi = u_\lambda^-$ in the weak equation $\langle DJ^+(u_\lambda), \psi \rangle = 0$, it holds that $u_\lambda^- = 0$ and thus u_λ is a nonnegative solution to (1.2). To show now that u_λ satisfies (1.8) notice that the mountain pass lemma entails

$$J^+(u_\lambda) \leq c_\lambda := \max_{[0,e_\lambda]} J,$$

where $c_\lambda = O(\lambda^{-q/(p-q)})$ as $\lambda \to \infty$. Thus, (1.8) is a consequence of

$$J^+(u) = \frac{1}{p} \langle DJ^+(u), u \rangle + \left(\frac{1}{q} - \frac{1}{p} \right) \int_\Omega |u|^q \leq c_\lambda,$$

which holds for $u = u_\lambda$.

To show ii) consider an arbitrary sequence u_n of solutions to (1.2), each u_n corresponding to λ_n and $\lambda_n \to 0$. It holds that $\|u_n\|_\infty \to \infty$. In fact, if otherwise $\|u_n\|_\infty = O(1)$ then it follows by arguing as in the proof of Theorem 1 that $u_n \to 0$ in $C^{1,\beta}(\overline{\Omega})$. If $t_n = \|u_n\|_{1,p}$ and $u_n = t_n v_n$ we get

$$\int_\Omega |\nabla v_n|^p + \frac{1}{t_n^{p-q}} \int_\Omega |v_n|^q = \lambda_n \int_{\partial\Omega} |v_n|^p. \tag{4.24}$$

Extracting a subsequence we obtain that $v_n \to v$ weakly in $W^{1,p}(\Omega)$. Observe that (4.24) implies in particular $\|\nabla v_n\|_p \to 0$, so that v is a constant and the convergence $v_n \to v$ is in $W^{1,p}(\Omega)$. Hence v is nontrivial and we arrive at a contradiction with (4.24) since $q < p$ and $t_n \to 0$. Therefore, $\|u_n\|_\infty \to \infty$.

We set $u_n = t_n w_n$ where now $t_n = \|u_n\|_\infty$ to conclude (as in the proof of Theorem 1) that $w_n \to \pm 1$ in $C^{1,\beta}(\Omega)$. Relation (1.9) easily follows from the equality

$$\int_\Omega \varphi_q(w_n) = \lambda_n t_n^{p-q} \int_{\partial\Omega} \varphi_p(w_n).$$

This concludes the proof. □

Remark 11. Existence of a nonnegative solution in Theorem 5 can be alternatively achieved by the approach used in [13] for the case $p = 2$. Namely, by obtaining an absolute minimizer of functional J in the manifold $\{u \in W^{1,p}(\Omega) : \int_\Omega |u|^q = 1\}$.

Proof of Theorem 6. We use Theorem 8 and observe that it has been already shown in the proof of Theorem 5 that the functional J fulfills conditions J-i) and ii) while J-iii) is obvious. To check J-iv) we choose, as in Theorem 2, $X_n =$ span $\{\phi_1, \ldots, \phi_n\}$, the ϕ_i's being the sequence of normalized eigenfunctions associated to the Stekolv eigenvalues λ_i. Then, inequality (3.12) implies that

$$J(u) \leq \frac{1}{2}(\lambda_n - \lambda)|\bar{t}|_2^2 + C|\bar{t}|_2^q, \quad u = \sum_{i=1}^n t_i\phi_i \in X_n.$$

By choosing $K_n = \{u \in X_n : |\bar{t}|_2 = R\}$ and $R > 0$ large, condition J-iv) holds for $\lambda > \lambda_n$. Then Theorem 8 implies that

$$0 < C_1\varepsilon_0^p \leq c_n = \inf_{K \in \mathcal{C}_n} \max_K J, \tag{4.25}$$

defines a critical value for J with at least two nontrivial associated critical points. This shows i).

The corresponding proof for ii) fits the same pattern. In fact, observe that J-i) and ii) were shown for $p > 1$ arbitrary meanwhile, as in Theorem 4,

$$J(u) \leq \frac{1}{p}\left(\frac{n^p}{B_n}\lambda_{n,p} - \lambda\right)|\bar{t}|_2^p + C|\bar{t}|_2^q < 0, \quad u = \sum_{i=1}^n t_i\phi_{i,p} \in X_{n,p},$$

with $X_{n,p} =$ span $\{\phi_{1,p}, \ldots, \phi_{n,p}\}$. Hence, J-iv) holds by taking $K_n = \{u \in X_{n,p} : |\bar{t}|_2 = R\}$, $R > 0$ large and

$$\lambda > \frac{n^p}{B_n}\lambda_{n,p}.$$

Analogously as before, the value $c_n > 0$ defined in the previous expression (4.25) also furnishes a pair of nontrivial solutions to (1.2). □

Proof of Theorem 7. Using the same notation as in the proof of Theorem 3, problem (1.1) can be written as equation (3.14),

$$u - (\lambda + 1)B(u) + A(\varphi_q(u)) = 0,$$

where we look for solutions $u \in H^1(\Omega)$. However, for our present purposes u must be searched in a more regular space. In fact, according to Schauder theory [19] the operators $A : C^\alpha(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega}), B : C^{1,\alpha}(\partial\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ are continuous (recall that Ω is now assumed to be $C^{2,\alpha}$). Hence $B : C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ is compact (here B is composed with the trace operator). Since H^1 solutions u to (3.14) lie in $C^1(\overline{\Omega})$ then such solutions belong to $C^{2,\beta}(\overline{\Omega})$ with $\beta = \min\{\alpha, q - 1\}$. In particular, equation (3.17) can be regarded in $C^2(\overline{\Omega})$.

In view of Schauder estimates,

$$\|A(\varphi_q(u))\|_{C^2(\overline{\Omega})} \leq \|A(\varphi_q(u))\|_{C^{2,\beta}(\overline{\Omega})} \leq C\|\varphi_q(u)\|_{C^\beta(\overline{\Omega})} \leq C\|u\|^{q-1}_{C^1(\overline{\Omega})},$$

and so the nonlinear term in (3.17) is $o(\|u\|_{C^1(\overline{\Omega})})$ as $\|u\|_{C^2(\overline{\Omega})} \to \infty$. In addition, the nonlinear operator $K(u) = \|u\|^{2(2-q)}_{C^2(\overline{\Omega})} A(\varphi_q(u))$, defined as 0 at $u = 0$, is continuous and compact in $C^2(\overline{\Omega})$. Therefore, the existence of solutions bifurcating from infinity follows from Theorem 1.6 in [26].

Let now (λ_k, u_k) be a sequence of solutions such that $\lambda_k \to \bar{\lambda}$, $s_k = \|u_k\|_\infty \to \infty$. Extracting a subsequence if necessary one obtains that $\tilde{u}_k = s_k^{-1} u_k \to \bar{\phi}$ in $C^1(\overline{\Omega})$, with $\bar{\phi}$ an eigenfunction associated to $\bar{\lambda}$. Setting Y the eigenspace of $\bar{\lambda}$ and π its associated orthogonal projection, $v_k = \pi(\tilde{u}_k)$, $w_k = (I - \pi)(\tilde{u}_k)$ (see the proof of Theorem 3) it follows that

$$\frac{\lambda_k - \bar{\lambda}}{\bar{\lambda} + 1}(v_k, \bar{\phi}) = s_k^{q-2} \int_\Omega \varphi_q(v_k + w_k)\bar{\phi}. \tag{4.26}$$

Since both the right hand side and $(v_k, \bar{\phi})$ become positive for large k one concludes that no sequences of bifurcated solutions exist so that $\lambda_k \leq \bar{\lambda}$ for all k. This proves i).

In case that $\bar{\lambda}$ is simple, Corollary 1.8 in [26] provides us the existence of a closed connected set (λ, u) of bifurcated solutions such that $u = s(\bar{\phi} + w_1)$, $s \geq s_0 > 0$, with $\lambda = \bar{\lambda} + o(1)$ and $w_1 = o(1)$ in $C^2(\overline{\Omega})$ as $s \to \infty$ and where $\bar{\phi}$ is an associated eigenfunction so that $\|\bar{\phi}\|_\infty = 1$. Relation (4.26) can now be written as

$$(\lambda_k - \bar{\lambda})\left(\int_{\partial\Omega} |\bar{\phi}|^2 + o(1)\right) = s_k^{q-2}\left(\int_\Omega |\bar{\phi}|^q + o(1)\right),$$

which proves (1.11). □

5. Spatial dependent reactions

We are next dealing with the variable coefficient version (1.12) of problem (1.2). Namely,

$$\begin{cases} \Delta_p u = a(x)\varphi_q(u), & x \in \Omega, \\ \mathcal{B}_p(u) = \lambda\varphi_p(u), & x \in \partial\Omega, \end{cases}$$

where we are assuming that $a \in C(\overline{\Omega})$ is *nonnegative* while $\Omega^+ := \{x \in \Omega : a(x) > 0\}$ is a nonempty $C^{1,\alpha}$ domain (in particular $a \not\equiv 0$), being $\Omega^+ = \Omega$ possible. However, in the relevant case, $\Omega^+ \subsetneq \Omega$ and $\Gamma_0 := \partial\Omega^+ \cap \Omega$ is a nonempty open part of $\partial\Omega^+$. In that case, it will be always assumed that Γ_0 is in addition closed i. e. $\overline{\Gamma}_0 = \Gamma_0$. It will be said for brevity that the function a satisfies hypothesis (H) when it fulfills all these conditions.

Observe that Γ_0 consists of the union of a finite number of components which define closed $C^{1,\alpha}$ manifolds. Therefore, if $\Omega_0 := \Omega \setminus \overline{\Omega^+} \neq \emptyset$ then Ω_0 is a $C^{1,\alpha}$ domain whose boundary is described as $\partial\Omega_0 = \Gamma_0 \cup \Gamma_1$ with $\Gamma_1 = \partial\Omega_0 \cap \partial\Omega$. Notice that $\Gamma_0 \cap \Gamma_1 = \emptyset$ while $\Gamma_1 = \emptyset$ if $\partial\Omega \subset \partial\Omega^+$. To simplify the exposition, we are also assuming that both Ω_0 and Γ_1 are connected and such features will be added to hypothesis (H) (see [12,13] for further insights and several possible configurations of Ω and Ω^+ allowed by condition (H)).

Theorem 12. *Assume that $a \in C(\overline{\Omega})$ is nonnegative and satisfies (H) while*

$$q > p > 1.$$

Then, problem (1.12) satisfies the following properties.

A) *If $\overline{\Omega}_0 \subset \Omega$ and so $\Gamma_1 = \emptyset$ then problem (1.12) exhibits essentially the same features as (1.2) in the superlinear regime. Namely, for all $\lambda > 0$ it possesses a unique pair $\pm u_\lambda \in C^{1,\beta}(\overline{\Omega})$ of one-signed solutions, $u_\lambda > 0$ in $\overline{\Omega}$. The solution u_λ bifurcates from zero at $\lambda = 0$ with*

$$u_\lambda = \bar{a}^{-\frac{1}{q-p}} \left(\frac{|\partial\Omega|}{|\Omega|} \right)^{\frac{1}{q-p}} \lambda^{\frac{1}{q-p}} (1 + o(1)), \tag{5.27}$$

as $\lambda \to 0$ in $C^{1,\beta}(\overline{\Omega})$, with $\bar{a} = \frac{1}{|\Omega|} \int_\Omega a$. Furthermore, $u_\lambda \to U$ in $C^{1,\beta}(\Omega)$ as $\lambda \to \infty$ where U is the minimal solution to the singular problem

$$\begin{cases} \Delta_p U = a(x)\varphi_q(U), & x \in \Omega, \\ U = \infty, & x \in \partial\Omega. \end{cases} \tag{5.28}$$

B) *If, on the contrary, $\Gamma_1 = \partial\Omega_0 \cap \partial\Omega \neq \emptyset$ then solutions to (1.12) exhibiting one sign in Ω are only possible when*

$$0 < \lambda < \sigma_{1,p}, \tag{5.29}$$

where $\sigma = \sigma_{1,p}$ is the principal eigenvalue to the problem

$$\begin{cases} \Delta_p u = 0 & x \in \Omega_0 \\ u = 0 & x \in \Gamma_0 \\ \mathcal{B}_p(u) = \lambda\varphi_p(u) & x \in \Gamma_1. \end{cases} \tag{5.30}$$

Moreover, for every λ satisfying (5.29), (1.12) admits a unique pair $\pm u_\lambda \in C^{1,\beta}(\overline{\Omega})$ of one-signed solutions, meanwhile u_λ is positive in $\overline{\Omega}$ and bifurcates from the trivial solution at $\lambda = 0$ with the profile (5.27).

Remark 12. a) Existence, uniqueness and simplicity of a principal eigenvalue $\sigma_{1,p}$ to (5.30) can be achieved by similar methods as those employed in [12] (see Theorem 6 there). For future reference we set $W^{1,p}_{\Gamma_0}(\Omega_0)$ the subspace of all those functions $u \in W^{1,p}(\Omega_0)$ which vanish on Γ_0. In addition, a variational expression of $\sigma_{1,p}$ is

$$\sigma_{1,p} = \inf \frac{\int_{\Omega_0} |\nabla u|^p}{\int_{\Gamma_1} |u|^p},$$

where $u \in W^{1,p}(\Omega_0) \setminus \{0\}$.

b) The precise asymptotic profile of solution u_λ as $\lambda \to \sigma_{1,p}$ in case B) of Theorem 12 can be described. Specifically, it can be proved that $u_\lambda \to \infty$ uniformly in $\Omega_0 \cup \Gamma_0$ as $\lambda \to \sigma_{1,p}-$ while $u_\lambda \to U$ in $C^{1,\beta}(\Omega^+ \cup \Gamma^+)$ as $\lambda \to \sigma_{1,p}-$, where $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$ and U satisfies a suitable boundary value problem in Ω^+ (see [12] for the linear diffusion case $p = 2$). Details are omitted in order to keep this work within a reasonable size.

A key result for the proof of Theorem 12 and further purposes is the following.

Lemma 13. *Assume $a \in C(\overline{\Omega})$ satisfies condition (H) and $\omega = \sigma_{1,p}$ if $\Gamma_1 \neq \emptyset$, $\omega = \infty$ otherwise. Then for every $\varepsilon > \frac{1}{\omega}$ there exists $C = C(\varepsilon)$ such that*

$$\int_{\partial\Omega} |u|^p \leq \varepsilon \int_\Omega |\nabla u|^p + C \int_\Omega a(x)|u|^p, \tag{5.31}$$

for every $u \in W^{1,p}(\Omega)$.

Proof. If the assertion does not hold then there exist $\varepsilon_0 > \frac{1}{\omega}$, and a sequence $u_n \in W^{1,p}(\Omega)$ such that

$$\int_{\partial\Omega} |u_n|^p > \varepsilon_0 \int_\Omega |\nabla u_n|^p + n \int_\Omega a(x)|u|^p,$$

for all n. Setting as usual $u_n = t_n v_n$, $t_n^p = \int_{\partial\Omega} |u_n|^p$ then, modulus a subsequence, $v_n \to v$ weakly in $W^{1,p}(\Omega)$ where $v = 0$ in Ω^+ and $\int_{\partial\Omega} |v|^p = 1$. If $\Gamma_1 = \emptyset$ then $\partial\Omega \subset \partial\Omega^+$ and $v = 0$ on $\partial\Omega$ what is not possible. On the contrary, if $\Gamma_1 \neq \emptyset$ then $\int_{\Gamma_1} |v|^p = 1$ since $v = 0$ on Γ^+ (if Γ^+ is non void). Thus

$$\sigma_{1,p} \leq \frac{\int_{\Omega_0} |\nabla v|^p}{\int_{\Gamma_1} |v|^p} \leq \frac{1}{\varepsilon_0},$$

contradicting the assumption on ε_0. □

Let

$$J_a(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{\lambda}{p} \int_{\partial\Omega} |u|^p + \frac{1}{q} \int_\Omega a(x)|u|^q$$

be the energy functional associated to (1.12) where $u \in W^{1,p}(\Omega)$.

Lemma 14. *The functional J_a is coercive provided $0 < \lambda < \omega$ and $q > p$. Furthermore, J_a satisfies PS under that range for λ and both when $q > p$ and $1 < q < p$.*

Proof. Observe that

$$J_a(u) \geq \frac{1}{p}(1 - \lambda\varepsilon) \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega a|u|^q - C \int_\Omega a|u|^p,$$

$\varepsilon > 0$ being chosen so that $\lambda < \frac{1}{\varepsilon} < \omega$. Thus,

$$J_a(u) \geq \frac{1}{p}(1 - \lambda\varepsilon) \int_\Omega |\nabla u|^p + C \int_\Omega a|u|^p - K,$$

for a certain constant K, provided that $q > p$. Thus, coercivity follows from the fact that the first two terms in the right hand side are comparable to $\|\cdot\|_{1,p}^p$.

To show that J_a verifies PS it is enough to prove that u_n is bounded in $W^{1,p}(\Omega)$ whenever $J(u_n) = O(1)$. Once this is obtained, it suffices to proceed as in the case $a = 1$ (proof of Theorem 2). On the other hand, the former assertion follows from coercivity when $q > p$.

As for the case $1 < q < p$ we choose a sequence $u_n \in W^{1,p}(\Omega)$ satisfying $J_a(u_n) = O(1)$, $DJ_a(u_n) = o(1)$ in $(W^{1,p}(\Omega))^*$ and, for the moment, suppose that u_n is unbounded in $W^{1,p}(\Omega)$. Keeping the notation for v_n introduced in the first part of the proof of Theorem 5 we also conclude that

$$\int_\Omega a(x)|v_n|^q = o(t_n^{-(q-1)}),$$

where $t_n \to \infty$. Since v_n is bounded in $W^{1,p}(\Omega)$ then, passing through a subsequence, $v_n \to v$ in $L^p(\Omega) \cap L^p(\partial\Omega)$ with $\int_{\partial\Omega} |v|^p = 1$ and $v = 0$ in Ω^+. This is impossible if $\Gamma_1 = \emptyset$ so assume that $\omega < \infty$. Then, $v \in W^{1,p}_{\Gamma_0}(\Omega_0)$ with

$$\int_{\Omega_0} |\nabla v|^p - \lambda \int_{\Gamma_1} |v|^p \le 0,$$

and $\int_{\Gamma_1} |v|^p = 1$, which is incompatible with $\lambda < \sigma_{1,p}$. Thus, the sequence u_n must be bounded in $W^{1,p}(\Omega)$. □

Arguing as in Proposition 10 one finds that weak solutions $u \in W^{1,p}(\Omega)$ to (1.12) lie also in $L^\infty(\Omega)$, with a uniform bound that only depends on p, Ω, λ and $\|u\|_{L^1(\Omega)}$ but not on q, nor on $a(x)$. Therefore, using the truncation $h_m(u)$ introduced in Sect. 2.3 with the same proof we obtain our next auxiliary result. It allows a complete freedom in the choice of exponent $q > p$ in (1.12).

Lemma 15. *Under the preceding assumptions on $a \in C(\overline{\Omega})$, there exists m_0 such that problem*

$$\begin{cases} \Delta_p u = a(x)h_m(u), & x \in \Omega, \\ \mathcal{B}_p(u) = \lambda\varphi_p(u) & x \in \partial\Omega, \end{cases} \tag{5.32}$$

exhibits the same solutions as (1.12) for all $m \ge m_0$ provided that $0 < \lambda < \omega$.

We can now proceed to show Theorem 12.

Proof of Theorem 12. In both cases A) and B) we find that J_a admits, due to coercivity, a positive global minimizer $u \in C^{1,\beta}(\overline{\Omega})$ for every $0 < \lambda < \omega$. Regarding uniqueness, if $v \in W^{1,p}(\Omega) \setminus \{0\}$ is a further nonnegative solution then the strong maximum principle yields $v > 0$ in $\overline{\Omega}$. We then introduce the integral

$$I_1 := \int_\Omega |\nabla v|^p - |\nabla u|^{p-2}\nabla u \nabla \left(\frac{v^p}{u^{p-1}}\right),$$

and its symmetric counterpart

$$I_2 := \int_\Omega |\nabla u|^p - |\nabla v|^{p-2}\nabla v \nabla \left(\frac{u^p}{v^{p-1}}\right).$$

The generalized Picone's inequality introduced in [1] states that both integrals are nonnegative. Moreover, v must be a scalar multiple of u in the case that one of them vanishes. By the weak form of (1.12) and a suitable choice of test functions we find that

$$I_1 + I_2 = \int_\Omega a(x)(v^p - u^p)(u^{q-p} - v^{q-p}) \le 0.$$

Accordingly, $I_1 = I_2 = 0$ and so, say $v = cu$. However, since

$$(c^{q-p} - 1)(c^p - 1) \int_\Omega a(x)u^q = 0,$$

then $c = 1$, as desired.

That u_λ bifurcates from zero at $\lambda = 0$ with the asymptotic behavior (5.27) is shown exactly as in Theorem 1.

On the other hand, that u_λ converges in case A) to the minimal solution U of the singular problem (5.28) is obtained by adapting the proof of Theorem 4 in [12] to the p-Laplacian framework. It should be stressed that the sole existence of such solution U is by no means straightforward when $\overline{\Omega}_0 \subset \Omega$ is non empty.

To conclude the proof let us show that no positive solutions to (1.12) are possible when $\Gamma_1 \neq \emptyset$ and $\lambda \geq \sigma_{1,p}$. In fact, if such solution u exists it becomes positive in Ω. Setting $\phi_1 \in W^{1,p}_{\Gamma_0}(\Omega_0)$ a positive eigenfunction associated to $\sigma_{1,p}$ in (5.30) and by choosing a convenient test function in (1.12) (ϕ_1 is regarded as zero in Ω^+) we obtain

$$\int_{\Omega_0} |\nabla \phi_1|^p - |\nabla u|^{p-2}\nabla u \nabla \left(\frac{\phi_1^p}{u^{p-1}}\right) = (\sigma_{1,p} - \lambda) \int_{\Gamma_1} \phi_1^p.$$

Being the first integral nonnegative, the only possibility is $\lambda = \sigma_{1,p}$. This in turn means that u vanishes on Γ_0 what is not possible. □

Concerning global existence of solutions when $\lambda > \lambda_n$, λ_n the n-th Steklov eigenvalue, and the corresponding result for $p \neq 2$, we have the following theorem. Recall that we define $\omega = \omega(\Omega_0)$ as $\sigma_{1,p}$ if $\Gamma_1 \neq \emptyset$, $\omega = \infty$ otherwise.

Theorem 16. *Assume $\Omega \subset \mathbb{R}^N$ is a $C^{1,\alpha}$ bounded domain and that $a \in C(\overline{\Omega})$ satisfies (H) and $q > p$. Then,*

i) *In the case $p = 2$, the conclusion of Theorem 2 holds provided*

$$\lambda_n < \lambda < \omega.$$

ii) *If $p \neq 2$, the assertions of Theorem 4 remain true when*

$$\frac{n^p}{B_n}\lambda_{1,p} < \lambda < \omega,$$

where the constant B_n is just the constant introduced in that statement.

Theorem 16, whose proof is entirely analogous to the corresponding ones contained in Sect. 3, poses the question on how large could the gap between λ_n and ω be in terms of the size of Ω_0. Our next result says that the smaller the measure of Ω_0, the larger the gap.

Lemma 17. *Let $\Gamma_1 \subset \partial\Omega$ be a fixed component of the boundary $\partial\Omega$ and define, for small $\delta_0 > 0$, $U = \{x \in \mathbb{R}^N : dist\ (x, \Gamma_1) < \delta_0\}$. If $\Omega_0 \subset U^- := U \cap \Omega$ is a smooth domain so that $\Gamma_1 \subset \partial\Omega_0$ and $\sigma_{1,p}(\Omega_0)$ is the first eigenvalue to (5.30) then $\sigma_{1,p}(\Omega_0) \to \infty$ as $|\Omega_0| \to 0+$.*

Proof. As a first remark, there exists a constant C_1, only depending on Γ_1, such that

$$\int_{\Gamma_1} |u|^p \le C_1 \int_{U^-} |u|^{p-1}|\nabla u| \le \frac{C_1\varepsilon^p}{p}\int_{U^-}|\nabla u|^p + \frac{C_1\varepsilon^{-p'}}{p'}\int_{U^-}|u|^p, \quad (5.33)$$

for every $u \in W^{1,p}(U^-)$ vanishing on $\partial U^- \setminus \Gamma_1$ and every prefixed $\varepsilon > 0$. On the other hand, a Lipschitz mapping $T : U \to U^-$ can be found so that $E : W^{1,p}(U^-) \to W^{1,p}(U)$, defined on smooth functions $u \in C^1(\overline{U})$ as $E(u)(x) = u(T(x))$ for $x \in U^+ := U \cap (\mathbb{R}^N \setminus \overline{\Omega})$, gives rise to a linear continuous extension operator.

Assume now that $\Omega_0 \subset U^-$ is as in the statement and $u \in W^{1,p}_{\Gamma_0}(\Omega_0)$ with $\Gamma_0 = \partial\Omega_0 \cap U^-$. Then, $\overline{u} = E(u) \in W^{1,p}_0(U)$ and has its support in $\Omega_0 \cup \Gamma_1 \cup \Omega_0^*$ where $\Omega_0^* = T^{-1}(\Omega_0)$ is the "reflection" of Ω_0 with respect to Γ_1. Thus,

$$\int_{\Omega_0} |u|^p \le \int_{\Omega_0\cup\Omega_0^*} |\overline{u}|^p \le \left(\frac{|\Omega_0| + |\Omega_0^*|}{\omega_N}\right)^{\frac{p}{N}} \left(\int_{\Omega_0}|\nabla u|^p + \int_{\Omega_0^*}|\nabla \overline{u}|^p\right), \quad (5.34)$$

where $\omega_N = |B_1(0)|$ and Poincare's inequality has been used (see the version of such inequality given by equation (7.44) in [17]). Since $|\Omega_0^*| \le C_1|\Omega_0|$ for a constant C_1 only depending on Γ_1 then we get from (5.34) that

$$\int_{\Omega_0} |u|^p \le C_1 |\Omega_0|^{\frac{p}{N}} \int_{\Omega_0} |\nabla u|^p,$$

for all $u \in W^{1,p}_{\Gamma_0}(\Omega_0)$. By combining such estimate with (5.33) under the choice $\varepsilon = |\Omega_0|^\theta$, $\theta > 0$, we obtain

$$\int_{\Gamma_1} |u|^p \le \frac{C_1}{p}\Phi(|\Omega_0|)\int_{\Omega_0}|\nabla u|^p \quad u \in W^{1,p}_{\Gamma_0}(\Omega_0),$$

where $\Phi(|\Omega_0|) = |\Omega_0|^{\theta p} + (p-1)|\Omega_0|^{\frac{p}{N}-\theta p'} \to 0$ as $|\Omega_0| \to 0$ provided that θ has been chosen so that $0 < \theta < \frac{p-1}{N}$. The desired conclusion then follows from the estimate

$$\frac{p}{C_1\Phi(|\Omega_0|)} \le \sigma_{1,p}(\Omega_0),$$

in which, constant C_1 only depends on Γ_1. □

Theorem 16 only provides nontrivial solutions to (1.12) for values of $\lambda < \omega$. However, and as should be expected, bifurcation from the Steklov eigenvalues still remains true regardless the structure of a.

Theorem 18. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{1,\alpha}$ bounded domain while $a = a(x)$, $a \not\equiv 0$, is merely nonnegative and continuous in $\overline{\Omega}$. Assume $q > p$ and $p = 2$. Then, all Steklov eigenvalues $\lambda = \lambda_n$ provide bifurcation values for λ from which m pairs $\pm u_{k,n}(\lambda)$, $1 \le k \le m$, of nontrivial solutions arise (m being the multiplicity of λ_n). Bifurcation occurs from the right of λ_n and the remaining conclusions of Theorem 3 hold. In particular, the profile estimate (1.6) is true but with the coefficient $E_n = \int_{\partial\Omega}\phi^2/\int_\Omega a|\phi|^q$.*

Proof. We begin with the remark that we deal with small amplitude solutions to (1.12). Thus, it can be assumed that all such solutions satisfy $\|u\|_{H^1(\Omega)} \leq \varepsilon$ for some small $\varepsilon > 0$ and λ near certain eigenvalue λ_n. According to Proposition 10 those solutions satisfy a uniform bound $\|u\|_{L^\infty(\Omega)} \leq m_0$ for certain $m_0 > 0$. This means that when searching small amplitude solutions, (1.12) can be replaced by its truncated version (5.32) with $m \geq m_0$ without requiring further conditions either on a nor on the size of λ. Recall that such step is required in order to work with a properly defined C^1 energy functional.

Now the general lines of the proof of Theorem 3 can be repeated. A final caution must be taken when showing that bifurcation at $\lambda = \bar{\lambda}$ occurs from the right, that is, $\lambda > \bar{\lambda}$. If the contrary were true we would find, from the argument of the last part of that proof, a sequence of eigenfunctions

$$v_m = \sum_{i=1}^{n} t_i^{(m)} \bar{\phi}_i \in Y,$$

which converges to some eigenfunction $v \neq 0$ that now satisfies

$$\int_\Omega a|v|^q = 0.$$

This means that v vanishes identically at least in some ball $B \subset \Omega$. This is not possible since v is non trivial and harmonic in Ω. □

Remark 13. If $a \in C(\overline{\Omega})$, $a \not\equiv 0$, is two signed and $2 < q \leq \frac{2N}{N-2}$ (suppose for instance that $N \geq 3$) then the Steklov eigenvalues still are bifurcation values. However, bifurcated solutions near an eigenvalue $\bar{\lambda}$ with multiplicity n are parameterized by their $H^1(\Omega)$ norm, rather than by parameter λ. More precisely, to every small $\varepsilon > 0$ there correspond m pairs of solutions $\pm u_{k,n,\lambda_\varepsilon}$, $1 \leq k \leq m$, each of them solving (1.12) for $\lambda = \lambda_\varepsilon$, such that $\|u_{k,n,\lambda_\varepsilon}\|_{H^1(\Omega)} = \varepsilon$ and $\lambda_\varepsilon \to \bar{\lambda}$ as $\varepsilon \to 0+$ (see [24]).

Let us state now the results for problem (1.12) corresponding to the sublinear regimen $1 < q < p$. They are all gathered together in a single statement.

Theorem 19. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,\alpha}$ domain, $a \in C(\overline{\Omega})$ a nonnegative function satisfying (H) while*

$$1 < q < p.$$

Then the following properties hold.

i) *For every λ satisfying*

$$0 < \lambda < \omega,$$

there corresponds a pair $\pm u_\lambda \in C^{1,\beta}(\overline{\Omega})$ of one-signed solutions such that

$$u_\lambda = \bar{a}^{\frac{1}{p-q}} (|\partial\Omega|/|\Omega|)^{-\frac{1}{p-q}} \lambda^{-\frac{1}{p-q}} (\pm 1 + o(1)),$$

as $\lambda \to 0+$ where $\bar{a} = \frac{1}{|\Omega|} \int_\Omega a$.

ii) *In case $\Gamma_1 \neq \emptyset$ and so $\omega = \sigma_{1,p}$ then all possible one-signed solutions u to (1.12) for $\lambda \geq \sigma_{1,p}$ vanish on Ω_0. In particular, no nontrivial and one-signed solutions exist for $\lambda \geq \sigma_{1,p}$ provided $\overline{\Omega^+} \subset \Omega$.*

iii) *For $p = 2$, problem (1.12) possesses n pairs $\pm u_{k,n}(\lambda)$, $1 \leq k \leq n$, of nontrivial solutions for every*

$$\lambda_n < \lambda < \omega.$$

iv) *If $p \neq 2$ the same conclusion as in* iii) *holds provided*

$$\frac{n^p}{B_n}\lambda_{n,p} < \lambda < \omega,$$

where B_n is the constant introduced in Theorem 4.

v) *Assume that $p = 2$, Ω is class $C^{2,\alpha}$ while $a \in C^\alpha(\overline{\Omega})$ is a nonnegative function without further requirements. Then problem (1.12) satisfies the conclusions of Theorem 7. In particular, bifurcation from infinity occurs at all Steklov eigenvalues $\bar{\lambda}$ with odd multiplicity. Such bifurcation occurs at $\lambda > \bar{\lambda}$.*

Remarks 14. a) As shown in the case $p = 2$ (cf. [13]) nontrivial and nonnegative solutions to (1.12) when $\lambda > \sigma_{1,p}$ are still possible if $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$ is nonempty and λ is large. Observe that *ii)* in Theorem 19 requires $\partial\Omega^+ \subset \Omega$.

b) The nonnegativity condition on a in v) is needed for the bifurcation to take place at the right of $\bar{\lambda}$. Nevertheless, bifurcation still occurs if $a \in C^\alpha(\overline{\Omega})$ changes sign.

Sketch of the proof of Theorem 19. Since it has already been proved in Lemma 14 that the functional J_a satisfies PS, we show that J_a has a strict local minimum at $u = 0$. Due to the presence of a, the argument in the proof of Theorem 5 must be modified. Thus, suppose on the contrary that there exists a sequence $u_n \in W^{1,p}(\Omega) \setminus \{0\}$ so that $u_n \to 0$ with $J_a(u_n) \leq 0$. Then we both obtain that $t_n^p := \int_{\partial\Omega} |u_n|^p \neq 0$ and that $t_n \to 0$. Setting $u_n = t_n v_n$ we observe that

$$\int_\Omega |\nabla v_n|^p - \lambda + \frac{p}{q} t_n^{q-p} \int_\Omega a|v_n|^q \leq 0$$

for every n. This implies that $v_n \to v$ weakly in $W^{1,p}(\Omega)$ with $\int_{\partial\Omega} |v|^p = 1$ and

$$\int_\Omega a|v|^q = 0.$$

Thus $v = 0$ in Ω^+. This is not possible if $\Gamma_1 = \emptyset$ since then we would have $\partial\Omega \subset \partial\Omega^+$ and so $v = 0$ on $\partial\Omega$. On the other hand, $v \in W^{1,p}_{\Gamma_0}(\Omega_0)$ if $\Gamma_1 \neq \emptyset$. From the previous integral inequality we find

$$\int_{\Omega_0} |\nabla v|^p - \lambda \int_{\Gamma_1} |v|^p \leq 0,$$

which in turns says that $\lambda \geq \sigma_{1,p}$ contradicting the assumptions. Therefore, $u = 0$ is a strict local minimum.

To complete the description of the local geometry of J_a near zero, let us observe that there also exist positive constants α, ρ such that condition *J*-*ii*) in Sect. 2.1 holds. This follows from [16] (see Corollary 1.6). After these preliminaries, the remaining part of the proof of $i)$ follows the corresponding one for the case $a = 1$.

Let us prove $ii)$. Suppose u is a nonnegative solution corresponding to $\lambda \geq \sigma_{1,p}$. If $u \neq 0$ in Ω_0 then, in view of the p-harmonic character of u in Ω_0 and the strong maximum principle, it follows that $u > 0$ in $\overline{\Omega}_0$. We now argue as in the uniqueness part in Theorem 12 and choose ϕ a positive eigenfunction associated to $\sigma_{1,p}$ in Ω_0. Then we get

$$0 \leq \int_{\Omega_0} |\nabla \phi|^p - |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\phi^p}{u^{p-1}} \right) = (\sigma_{1,p} - \lambda) \int_{\Gamma_1} \phi^p.$$

This can only happen when $\lambda = \sigma_{1,p}$, but even in this case u would be an eigenfunction which is also impossible since u never vanishes on Γ_0. Therefore, $u = 0$ in Ω_0. On the other hand, $-\Delta_p u \leq 0$ in Ω^+. If, in addition, $\overline{\Omega^+} \subset \Omega$ then $u \leq 0$ on $\partial\Omega^+$ and so u also vanishes in Ω^+. Thus $ii)$ is proved.

There are no further novelties in the proofs of $iii)$, $iv)$, $v)$ and therefore they will be omitted. □

Acknowledgements

Supported by Spanish Ministerio de Ciencia e Innovación and Ministerio de Economía y Competitividad under grant reference MTM2011-27998.

References

[1] Allegretto, W., Huang, Y.X.: A Piccone's identity for the p-Laplacian and applications. Nonlinear Anal. TMA **2**, 143–175 (1996)

[2] Amann, H.: Lusternik–Schnirelman theory and non-linear eigenvalue problems. Math. Ann. **199**, 55–72 (1972)

[3] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)

[4] Anane, A.: *Étude des valeurs propres et de la résonance pour l'operateur p-Laplacien*. Thèse de Doctorat, Université Libre de Bruxelles (1987).

[5] Arrieta, J., Pardo, R., Rodríguez-Bernal, A.: Bifurcation and stability of equilibria with asymptotically linear boundary conditions at infinity. Proc. R. Soc. Edinb. **137**, 225–252 (2007)

[6] Arrieta, J., Pardo, R., Rodríguez-Bernal, A.: Equilibria and global dynamics of a problem with bifurcation from infinity. J. Differ. Equ. **246**, 2055–2080 (2009)

[7] Arrieta, J., Pardo, R., Rodríguez-Bernal, A.: Infinite resonant solutions and turning points in a problem with unbounded bifurcation. Int. J. Bifurc. Chaos **20**, 2885–2896 (2010)

[8] Bandle, C.: Isoperimetric Inequalities and Applications. Pitman, London (1980)

[9] Fernández-Bonder, J., Rossi, J.D.: Existence results for the p-Laplacian with nonlinear boundary conditions. J. Math. Anal. Appl. **263**, 195–223 (2001)

[10] Fernández-Bonder, J., Rossi, J.D.: A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding. Publ. Mat. **46**, 221–235 (2002)

[11] García Azorero, J.P., Peral Alonso, I.: Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues. Comm. Partial Differ. Equ. **12**, 1389–1430 (1987)

[12] García-Melián, J., Rossi, J.D., Sabina de Lis, J.: A bifurcation problem governed by the boundary condition.I. NoDEA Nonlinear Differ. Equ. Appl. **14**, 499–525 (2007)

[13] García-Melián, J., Rossi, J.D., Sabina de Lis, J.: A bifurcation problem governed by the boundary condition. II. Proc. Lond. Math. Soc. **94**, 1–25 (2007)

[14] García-Melián, J., Rossi, J.D., Sabina de Lis, J.: Layer profiles of solutions to elliptic problems under parameter-dependent boundary conditions. Z. Anal. Anwend. **29**, 451–467 (2010)

[15] García-Melián, J., Rossi, J.D., Sabina de Lis, J.: Limit cases in an elliptic problem with a parameter-dependent boundary condition. Asymptot. Anal. **73**(3), 147–168 (2011)

[16] Ghoussoub, N.: Duality and perturbation methods in critical point theory. Cambridge University Press, Cambridge, (1993).

[17] Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Springer, Berlin, (1983)

[18] Henrot, A.: Extremum problems for eigenvalues of elliptic operators. Birkhäuser, Basel (2006)

[19] Ladyženskaja, O.A.; Ural'tseva, N.N.: Linear and quasilinear elliptic equations. Academic Press, New York (1968)

[20] Lieberman, G.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**, 1203–1219 (1988)

[21] Lindqvist, P.: A nonlinear eigenvalue problem. In: Topics in mathematical analysis, pp. 175–203. World Scientific Publishing, Hackensack (2008)

[22] Lions, P.L.: On the existence of positive solutions of semilinear elliptic equations. SIAM Rev. **24**, 441–467 (1982)

[23] Rabinowitz, P.H.: "Théorie du degré topologique et applications à des problémes aux limites non linéaires". Paris VI et CNRS, 1975

[24] Rabinowitz, P.H.: Minimax methods in critical point theory with applications to differential equations. In: CBMS Regional Conference Series in Mathematics, vol. 65. American Mathematical Society, Providence (1986)

[25] Rabinowitz, P.H.: A bifurcation theorem for potential operators. J. Funct. Anal. **25**, 412–424 (1977)

[26] Rabinowitz, P.H.: On bifurcation from infinity. J. Funct. Anal. **14**, 462–475 (1973)

[27] Struwe, M.: Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. Springer, Berlin (2008)

[28] Tolksdorff, P.: Regularity for a more general class of quasilinear elliptic equations. J. Differ. Equ. **51**, 126–150 (1984)

Jorge García-Melián and José C. Sabina de Lis
Departamento de Análisis Matemático
Universidad de La Laguna
C/. Astrofísico Francisco Sánchez s/n
38271 La Laguna
Spain
e-mail: josabina@ull.es

Jorge García-Melián
e-mail: jjgarmel@ull.es

Jorge García-Melián and José C. Sabina de Lis
Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atómica
Molecular y Fotónica
Facultad de Física
Universidad de La Laguna
C/. Astrofísico Francisco Sánchez s/n
38203 La Laguna
Spain

Julio D. Rossi
Departamento de Análisis Matemático
Universidad de Alicante
Ap. correos 99
03080 Alicante
Spain
e-mail: julio.rossi@ua.es

Received: 10 September 2012.
Revised: 26 February 2013.
Accepted: 22 August 2013.