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Limit cases in an elliptic problem with a parameter-dependent boundary condition

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Abstract. In this work we discuss existence, uniqueness and asymptotic profiles of positive solutions to the quasilinear problem

$$\begin{cases} -\Delta_p u + a(x)u^{p-1} = -u^r & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda u^{p-1} & \text{on } \partial\Omega \end{cases}$$

for $\lambda \in \mathbb{R}$, where $r > p-1 > 0$, $a \in L^\infty(\Omega)$. We analyze the existence of solutions in terms of a principal eigenvalue, and determine their asymptotic behavior both when $r \to p-1$ and when $r \to \infty$.

Keywords: linear and nonlinear eigenvalue problems, sub- and super-solutions, variational methods

1. Introduction

The aim of the present paper is to analyze some qualitative features exhibited by the positive solutions to

$$\begin{cases} -\Delta_p u(x) + a(x)u^{p-1}(x) = -u^r(x), & x \in \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu}(x) = \lambda u^{p-1}(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $r > p-1 > 0$, Ω is a class $C^{2,\alpha}$ bounded domain of $\mathbb{R}^N$ ($N \geqslant 2$), $0 < \alpha \leqslant 1$, and ν stands for the outward unit normal field on $\partial\Omega$. The operator Δ_p is the standard p-Laplacian, which is defined in the usual weak sense of $W^{1,p}(\Omega)$ as $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. In addition, it will be assumed throughout that $a \in L^\infty(\Omega)$. The main feature of problem (1.1) is its dependence on the parameter λ precisely in the boundary condition.

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Problem (1.1) was studied in [4] when $p = 2$ (in this case Δ_p is the usual Laplacian) with fixed $r > 1$ and $a = 0$. Under these conditions, it was shown there that this problem admits a unique positive solution $u_{r,\lambda}$ for every $\lambda > 0$, and no positive solutions when $\lambda \leqslant 0$. It was further shown that $u_{r,\lambda}$ is continuous and increasing as a function of λ, and its asymptotic behavior when $\lambda \to 0$ and $\lambda \to \infty$ was also completely elucidated (see [4] for additional features). However, as far as we know, the dependence of $u_{r,\lambda}$ on r has not yet been clarified. Thus, one of the objectives of this work is to analyze the variation of $u_{r,\lambda}$ with respect to r, especially in the extreme cases where $r \to 1+$ or $r \to \infty$. This study will be indeed extended to cover the more general problem (1.1).

To deal with the quasilinear problem (1.1), a number of auxiliary results must be developed. In particular, a study of the flux-type eigenvalue problem

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = \mu|u|^{p-2}u(x), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = \lambda|u|^{p-2}u(x), & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where λ is regarded as a parameter and it is assumed that $a \in L^\infty(\Omega)$. A number $\mu \in \mathbb{R}$ is said to be an eigenvalue to (1.2) if there exists $\phi \in W^{1,p}(\Omega)$, not vanishing identically in Ω, so that

$$\int_\Omega \left(|\nabla\phi|^{p-2}\nabla\phi\nabla\varphi + a(x)|\phi|^{p-2}\phi\varphi\right) \mathrm{d}x = \lambda\int_{\partial\Omega}|\phi|^{p-2}\phi\varphi\,\mathrm{d}S + \mu\int_\Omega|\phi|^{p-2}\phi\varphi\,\mathrm{d}x$$

for all $\varphi \in W^{1,p}(\Omega)$. In that case, ϕ is called an eigenfunction associated to μ.

Problem (1.2) has been studied in detail in [5] when $p = 2$, in which case it becomes

$$\begin{cases} -\Delta u(x) + a(x)u(x) = \mu u(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = \lambda u(x), & x \in \partial\Omega. \end{cases} \tag{1.3}$$

The next statement is the extension to problem (1.2) of the corresponding results obtained for (1.3) contained in [5] (a slightly more general version of (1.3) was in fact considered there).

Theorem 1. *Problem* (1.2) *admits, for every* $\lambda \in \mathbb{R}$, *a unique* principal eigenvalue $\mu = \mu_{1,p}$, *i.e., an eigenvalue with a nonnegative associated eigenfunction* $\phi \in W^{1,p}(\Omega)$. *It is given by the variational expression*

$$\mu_{1,p} = \inf_{u \in W^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega(|\nabla u|^p + a|u|^p)\,\mathrm{d}x - \lambda\int_{\partial\Omega}|u|^p\,\mathrm{d}S}{\int_\Omega|u|^p\,\mathrm{d}x}.$$

In addition, the following properties hold true.

(i) $\mu_{1,p}$ *is the unique principal eigenvalue.*
(ii) $\mu_{1,p}$ *is isolated and simple.*
(iii) *Every associated eigenfunction* $\phi_1 \in W^{1,p}(\Omega)$ *to* $\mu_{1,p}$ *satisfies* $\phi \in L^\infty(\Omega)$ *and furthermore* $\phi \in C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(U_\eta)$ *for certain* $\beta \in (0,1)$, $\eta > 0$, *with* $U_\eta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \eta\}$.
(iv) *As a function of* λ, $\mu_{1,p}$ *is concave, decreasing and verifies*

$$\lim_{\lambda\to-\infty}\mu_{1,p} = \lambda_{1,p}(a), \qquad \lim_{\lambda\to\infty}\mu_{1,p} = -\infty,$$

where $\lambda_{1,p}(a)$ *is the first Dirichlet eigenvalue of* $-\Delta_p u + a(x)|u|^{p-2}u$ *in* Ω.

Another auxiliary eigenvalue problem we will need is

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = 0, & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = \sigma |u|^{p-2}u(x), & x \in \partial\Omega, \end{cases} \tag{1.4}$$

which constitutes an extension to the p-Laplacian setting of the well-known Steklov problem (see [12] for a detailed analysis of the case $a = 0$). As a direct consequence of Theorem 1 the following statement holds true.

Theorem 2. *Problem* (1.4) *possesses a principal eigenvalue if and only if*

$$\lambda_{1,p}(a) > 0. \tag{1.5}$$

Furthermore:

(i) *Provided that* (1.5) *is satisfied*, (1.4) *admits a unique principal eigenvalue $\sigma_{1,p}$ which is isolated and simple. In addition*,

$$\operatorname{sign} \sigma_{1,p} = \operatorname{sign} \lambda_{1,p}^{\mathcal{N}}(a), \tag{1.6}$$

where $\lambda_{1,p}^{\mathcal{N}}(a)$ stands for the first Neumann eigenvalue of $-\Delta_p u + a(x)|u|^{p-2}u$ in Ω.

(ii) *Any eigenfunction $\psi \in W^{1,p}(\Omega)$ associated to $\sigma_{1,p}$ satisfies $\psi \in C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(U_\eta)$ for certain $\beta \in (0,1)$, $\eta > 0$, with $U_\eta = \{x\colon \operatorname{dist}(x \in \Omega, \partial\Omega) < \eta\}$.*

Remark 1. We will set $\sigma_{1,p} = -\infty$ when $\lambda_{1,p}(a) \leqslant 0$, for reasons that will become clear later on (see (1.8) in Theorem 4 and Remark 3).

The well-known sub- and super-solutions method is another tool that must be properly adapted to problem (1.1). A function $\overline{u} \in W^{1,p}(\Omega)$ is said to be a super-solution to problem

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = f(x,u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = g(x,u), & x \in \partial\Omega, \end{cases} \tag{1.7}$$

if

$$\int_\Omega \left(|\nabla \overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi + a(x)|\overline{u}|^{p-2}\overline{u}\varphi\right)\mathrm{d}x \geqslant \int_{\partial\Omega} g(x,\overline{u})\varphi\,\mathrm{d}S + \int_\Omega f(x,\overline{u})\varphi\,\mathrm{d}x$$

holds for all nonnegative $\varphi \in W^{1,p}(\Omega)$. Subsolutions are defined in a symmetric way. Of course, the existence of the integrals involving f and g is implicitly assumed.

In order to avoid the use of comparison, which is certainly a delicate issue when dealing with the p-Laplacian, the next statement furnishes a variational version of the method of sub- and super-solutions for problem (1.7) (cf. also [14]). Recall that a function $h\colon X \times \mathbb{R} \to \mathbb{R}$, (X,μ) a measure space, is a Carathéodory function if $h(\cdot,u)$ is measurable in X for all $u \in \mathbb{R}$ while $h(x,\cdot)$ is continuous in $\mathbb{R}$ for almost all $x \in X$.

Theorem 3. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying $|f(x,u)| \leqslant M$ and $|g(x,u)| \leqslant M$ if $(x,u) \in \Omega \times (-R,R)$ and $(x,u) \in \partial\Omega \times (-R,R)$, respectively, for arbitrary R, where $M = M(R)$. Suppose $\underline{u}$, $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ are a sub- and a super-solution to* (1.7) *so that $\underline{u} \leqslant \overline{u}$ a.e. in Ω. Then* (1.7) *admits a solution $u \in W^{1,p}(\Omega)$ verifying*

$$\underline{u} \leqslant u \leqslant \overline{u},$$

a.e. in Ω.

After these preliminary tools have been introduced, we can state a first group of results concerning problem (1.1).

Theorem 4. *Assume $\Omega \subset \mathbb{R}^N$ is a class $C^{2,\alpha}$ bounded domain and $r > p-1 > 0$. Then the following properties hold*:

(i) *Problem* (1.1) *admits a positive solution if and only if*

$$\lambda > \sigma_{1,p}, \tag{1.8}$$

where the value $\sigma_{1,p} = -\infty$ is allowed. When (1.8) *holds, the positive solution is unique, and it will be denoted by $u_{r,\lambda} \in W^{1,p}(\Omega)$.*

(ii) *The function $u_{r,\lambda}$ belongs to $C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(U_\eta)$ for a certain $\beta \in (0,1)$ and $\eta > 0$ small enough, where $U_\eta = \{x \in \Omega\colon \operatorname{dist}(x,\partial\Omega) < \eta\}$.*

(iii) *The mapping $\lambda \to u_{r,\lambda}$ is increasing and continuous with values in $C^1(\overline{\Omega})$. Moreover,*

$$\lim_{\lambda \to \sigma_{1,p}+} u_{r,\lambda} = 0 \tag{1.9}$$

in $C^{1,\beta}(\overline{\Omega})$ provided $\sigma_{1,p} > -\infty$. If $\sigma_{1,p} = -\infty$ then

$$\lim_{\lambda \to \sigma_{1,p}+} u_{r,\lambda} = \begin{cases} 0 & \text{if } \lambda_{1,p}(a) = 0, \\ w & \text{if } \lambda_{1,p}(a) < 0, \end{cases} \tag{1.10}$$

where $u = w(x)$ stands for the unique positive solution to

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = -u^r(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.11}$$

when $\lambda_{1,p}(a) < 0$.

(iv) *Let $u = U(x)$ be the minimal solution to the singular boundary value problem*

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = -u^r(x), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega. \end{cases} \tag{1.12}$$

Then,

$$\lim_{\lambda \to \infty} u_{r,\lambda} = U \tag{1.13}$$

in $C^1(\Omega)$.

We turn now to study the asymptotic behavior of the positive solution $u_{r,\lambda}$ to (1.1) both as $r \to (p-1)+$ and when $r \to \infty$. Let us begin with the former case and to this purpose notice that Theorem 1(iv) implies the existence of a value λ^* such that

$$\mu_{1,p}(\lambda^*) = -1,$$

provided that $\lambda_{1,p}(a) > -1$ ($\lambda_{1,p}(a)$ being the principal Dirichlet eigenvalue of $-\Delta_p u + a(x)|u|^{p-2}u$ in Ω). Observe that

$$\sigma_{1,p} < \lambda^*,$$

even in the case when $\sigma_{1,p} = -\infty$, while

$$0 < -\mu_{1,p}(\lambda) < 1 \quad \text{for } \sigma_{1,p} < \lambda < \lambda^*.$$

Of course, the inequality holds for all $\lambda < \lambda^*$ if $\sigma_{1,p} = -\infty$. Similarly,

$$-\mu_{1,p}(\lambda) > 1 \quad \text{if } \lambda > \lambda^*.$$

On the other hand,

$$-\mu_{1,p}(\lambda) > 1 \quad \text{for all } \lambda$$

in the complementary case $\lambda_{1,p}(a) \leqslant -1$ where the value λ^* does not exist.

Then we have the following theorem.

Theorem 5. *For $\lambda > \sigma_{1,p} \geqslant -\infty$, let $u = u_{r,\lambda}$ be the unique positive solution to problem* (1.1) *for $r > p-1$. Then,*

$$\Big(\sup_{\Omega} u_{r,\lambda}\Big)^{r-p+1} = -\mu_{1,p}(\lambda) + \mathrm{o}(1)$$

as $r \to (p-1)+$ while

$$u_{r,\lambda} = \Big(\sup_{\Omega} u_{r,\lambda}\Big)\big\{\phi_1(\lambda) + \mathrm{o}(1)\big\}$$

in $C^1(\overline{\Omega})$ as $r \to (p-1)+$, where $\phi_1(\lambda)$ stands for the positive eigenfunction associated to $\mu_{1,p}(\lambda)$ normalized so as $\sup_{\Omega} \phi_1(\lambda) = 1$. In particular

(a) *$u_{r,\lambda} \to 0$ uniformly in $\overline{\Omega}$ as $r \to (p-1)+$ if $\lambda < \lambda^*$ provided that $\lambda_{1,p}(a) > -1$. Moreover, for $\lambda = \lambda^*$ and $p = 2$ in problem* (1.1) *then*

$$u_{r,\lambda} \to A\phi_1(\lambda^*)$$

uniformly in Ω as $r \to (p-1)+$ where A is given by

$$A = \exp\left(-\frac{\int_{\Omega} \phi_1^2 \log \phi_1 \,\mathrm{d}x}{\int_{\Omega} \phi_1^2 \,\mathrm{d}x}\right). \tag{1.14}$$

(b) *$u_{r,\lambda} \to \infty$ uniformly in $\overline{\Omega}$ as $r \to (p-1)+$ either when $\lambda > \lambda^*$ if $\lambda_{1,p}(a) > -1$ or for all $\lambda \in \mathbb{R}$ provided $\lambda_{1,p}(a) \leqslant -1$.*

Note that in the previous theorem the case $\lambda = \lambda^*$ with $p \neq 2$ is left open.

As for the behavior of the solution $u_{r,\lambda}$ to (1.1) when $r \to \infty$ the first interesting conclusion is that for every $\lambda > \sigma_{1,p}$, $u_{r,\lambda}$ keeps uniformly bounded in Ω as $r \to \infty$. On the other hand, provided that coefficient $a = 0$ in (1.1) we achieve a better result. Namely, solutions become flat throughout the domain Ω as r increases.

Theorem 6. *Assume that $a = 0$ in problem* (1.1). *Then, for any $\lambda > \sigma_{1,p}$ we have $u_{r,\lambda} \to 1$ uniformly in $\overline{\Omega}$ as $r \to \infty$.*

It should be mentioned that a similar analysis for the logistic problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) - b(x)u^r(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.15}$$

which is somehow related to (1.1), was performed in [2,3]. However, the situation was substantially different there when $r \to \infty$, since the limit problem so obtained is of a free boundary type, mainly due to the Dirichlet condition. On the other hand, if $u = \tilde{u}_{r,\lambda}$ stands for the unique positive solution to (1.15) for $\lambda > \lambda_1^{\mathcal{D}}$ (the first Dirichlet eigenvalue of $-\Delta$ in Ω), an important feature in the analysis in [3] is the fact that

$$\Big(\sup_{\Omega} \tilde{u}_{r,\lambda}\Big)^{r-1}$$

remains bounded as $r \to \infty$. This follows easily from the boundary condition when $b > 0$ in $\overline{\Omega}$. This fact is in strong contrast with the next result.

Theorem 7. *Let $a \in L^\infty(\Omega)$. Then, for fixed $\lambda > \sigma_{1,p}$*

$$\phi_1(\lambda) \leqslant \varliminf_{r\to\infty} u_{r,\lambda} \leqslant \varlimsup_{r\to\infty} u_{r,\lambda} \leqslant 1,$$

where $\phi_1(\lambda)$ is the positive eigenfunction associated to $\mu_{1,p}(\lambda)$ normalized so that $\sup \phi_1(\lambda) = 1$. In particular,

$$\lim_{r\to\infty} \sup_{\Omega} u_{r,\lambda} = 1.$$

However, if either $a = 0$ or $a \in L^\infty(\Omega)$ is arbitrary but $\lambda > \sigma_1(|a|_\infty)$ in (1.1) *then*

$$\lim_{r\to\infty} \sup_{\Omega} (u_{r,\lambda})^{r-p+1} = \infty.$$

The rest of the paper is organized as follows: in Section 2 we analyze the eigenvalue problems (1.2) and (1.4). Section 3 is dedicated to develop the method of sub- and super-solutions for problem (1.7), that will be used here for the proof of Theorem 4. Finally, in Section 4 the asymptotic behavior of the positive solution to (1.1) as $r \to p-1$ and $r \to +\infty$ is considered.

2. Eigenvalue problems

In this section we perform the analysis of the eigenvalue problems (1.2) and (1.4). We begin with a fundamental result concerning the boundedness of eigenfunctions.

Lemma 8. *Let $\phi \in W^{1,p}(\Omega)$ be an eigenfunction associated to an arbitrary eigenvalue μ of* (1.2). *Then $\phi \in L^\infty(\Omega)$.*

Proof. Notice that we may assume $1 < p \leqslant N$, since otherwise $W^{1,p}(\Omega) \subset L^\infty(\Omega)$. Also, for the sake of simplicity we will only consider $p < N$, the case $p = N$ being handled in a similar way.

For $k > 0$ set $v = (\phi - k)^+$, $A_k = \{x \in \Omega\colon\ \phi(x) > k\}$. We show an estimate of the form

$$|v|_1 \leqslant C k^\delta |A_k|^{1+\varepsilon} \tag{2.1}$$

for every $k \geqslant k_0$ and certain positive constants $k_0, C, \delta, \varepsilon$ with $\delta \leqslant 1 + \varepsilon$, where $|v|_1 = |v|_{L^1(\Omega)}$.

By using v as a test function in the equation for ϕ we obtain

$$\begin{aligned} \int_\Omega \big(|\nabla v|^p + \varphi_p(\phi) v\big)\,\mathrm{d}x &\leqslant \lambda \int_{\partial\Omega} \varphi_p(\phi) v\,\mathrm{d}S + \big(\mu + |a|_\infty + 1\big) \int_\Omega \varphi_p(\phi) v\,\mathrm{d}x \\ &\leqslant C\left\{\int_{\partial\Omega} \varphi_p(\phi) v\,\mathrm{d}S + \int_\Omega \varphi_p(\phi) v\,\mathrm{d}x\right\}, \end{aligned} \tag{2.2}$$

where $\varphi_p(\phi) = |\phi|^{p-2}\phi$ and C will stand in the sequel for a positive constant independent of ϕ and k, not necessarily the same everywhere.

Next notice that $0 < v < \phi$ in the support of v and $\phi \leqslant v + k$, hence $\varphi_p(\phi) \leqslant C(v^{p-1} + k^{p-1})$. Thus (2.2) implies

$$|v|_{1,p}^p \leqslant C\left\{\int_{\partial\Omega} v^p\,\mathrm{d}S + k^{p-1}\int_{\partial\Omega} v\,\mathrm{d}S + \int_\Omega v^p\,\mathrm{d}x + k^{p-1}\int_\Omega v\,\mathrm{d}x\right\} \tag{2.3}$$

for all $k > 0$, where $|v|_{1,p} = |v|_{W^{1,p}(\Omega)}$.

On the other hand, we notice that, thanks to Hölder's and Sobolev's inequalities:

$$\int_\Omega v^p\,\mathrm{d}x \leqslant |A_k|^{p/N}\left(\int_\Omega v^{p^*}\,\mathrm{d}x\right)^{p/p^*} \leqslant C|A_k|^{p/N}\left(\int_\Omega |\nabla v|^p\,\mathrm{d}x + \int_\Omega v^p\,\mathrm{d}x\right),$$

where $p^* = \frac{Np}{N-p}$, and, since $|A_k| \to 0$,

$$\int_\Omega v^p\,\mathrm{d}x \leqslant C|A_k|^{p/N}\int_\Omega |\nabla v|^p\,\mathrm{d}x \tag{2.4}$$

for $k \geqslant k_0$ and certain positive k_0.

Furthermore, it is useful to recall that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\int_{\partial\Omega} |u|^p\,\mathrm{d}S \leqslant \varepsilon\int_\Omega |\nabla u|^p\,\mathrm{d}x + C(\varepsilon)\int_\Omega |u|^p\,\mathrm{d}x \tag{2.5}$$

for every $u \in W^{1,p}(\Omega)$ (see, for instance, [5], Lemma 6, for a proof when $p = 2$). This inequality combined with (2.4) yields

$$\int_{\partial\Omega} v^p \, \mathrm{d}S \leqslant \left(\varepsilon + C(\varepsilon)|A_k|^{p/N}\right) \int_{\Omega} |\nabla v|^p \, \mathrm{d}x \tag{2.6}$$

for $k \geqslant k_0$. Inequalities (2.3), (2.4) and (2.6) imply, taking ε sufficiently small,

$$|v|_{1,p}^p \leqslant Ck^{p-1}\{|v|_{1,\partial\Omega} + |v|_1\} \tag{2.7}$$

for $k \geqslant k_0$, where $|v|_{1,\partial\Omega} = |v|_{L^1(\partial\Omega)}$.

Observe now that, thanks to the immersion $W^{1,1}(\Omega) \subset L^1(\partial\Omega)$ and Hölder's inequality

$$|v|_{1,\partial\Omega} \leqslant C|v|_{W^{1,1}(\Omega)} \leqslant C|A_k|^{1-1/p}|v|_{1,p},$$

while the Sobolev immersion gives

$$|v|_1 \leqslant C|A_k|^{1-1/p^*}|v|_{1,p}. \tag{2.8}$$

Thus, from (2.7) we get

$$|v|_{1,p} \leqslant Ck\{|A_k|^{1/p} + |A_k|^{1/(p-1)(1-1/p^*)}\} \leqslant Ck|A_k|^{1/p}$$

for all $k \geqslant k_0$, since $\frac{1}{p} < \frac{1}{p-1}(1 - \frac{1}{p^*})$ and $|A_k| \to 0$. This inequality allows us to conclude, thanks to (2.8), that

$$|v|_1 \leqslant Ck|A_k|^{1+1/N} \tag{2.9}$$

for large k, which is the desired inequality.

Finally, when (2.9) is combined with [9], Chapter 2, Lemma 5.1, we obtain $\phi^+ \in L^\infty(\Omega)$, and since $-\phi$ is also an eigenfunction, the preceding argument also says that $\phi \in L^\infty(\Omega)$. □

Remark 2. Lemma 8 can be also shown by means of a Moser's iteration procedure following the ideas in [5] (see Lemma 5 there).

Proof of Theorem 1. To show the existence of a principal eigenvalue we borrow ideas from [5], Lemma 7. Thus, consider $\mathcal{M} := \{u \in W^{1,p}(\Omega)\colon \int_\Omega |u|^p = 1\}$, and the functional

$$J(u) = \int_\Omega \left(|\nabla u|^p + a(x)|u|^p\right) \mathrm{d}x - \lambda \int_{\partial\Omega} |u|^p \, \mathrm{d}S.$$

Inequality (2.5) implies that

$$J(u) \geqslant \left(1 - \varepsilon|\lambda|\right) \int_\Omega |\nabla u|^p \, \mathrm{d}x - \left(|a|_\infty + C(\varepsilon)|\lambda|\right) \int_\Omega |u|^p \, \mathrm{d}x$$

for all $u \in W^{1,p}(\Omega)$. This means that J is coercive in $\mathcal{M}$ and the direct method in the calculus of variations (see [14], Theorem 1.2) implies the finiteness of

$$\mu_{1,p} = \inf_{u \in W^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega (|\nabla u|^p + a(x)|u|^p)\,\mathrm{d}x - \lambda \int_{\partial\Omega} |u|^p\,\mathrm{d}S}{\int_\Omega |u|^p\,\mathrm{d}x}$$

and the existence of $\phi \in W^{1,p}(\Omega)$ such that the infimum is achieved at $u = \phi$. Since the infimum is also attained at $|\phi|$, it is easily checked that $|\phi|$ defines an eigenfunction associated to $\mu_{1,p}$, hence $\mu_{1,p}$ is a principal eigenvalue.

Next, let $\phi \in W^{1,p}(\Omega)$ be a nonnegative eigenfunction associated to $\mu_{1,p}$. Lemma 8 and Lieberman's regularity results [10] imply that $\phi \in C^{1,\beta}(\overline{\Omega})$ for a certain $0 < \beta < 1$ while the Strong Maximum Principle in [15] implies that $\phi > 0$ throughout $\overline{\Omega}$ together with $|\nabla\phi| > 0$ in some strip $U_\eta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \eta\}$. Then, the equation for ϕ becomes strictly elliptic in U_η and standard theory of quasilinear equations yields $\phi \in C^{2,\alpha}(U_\eta)$ (cf. [9]).

As a consequence of the preceding assertions it follows that every eigenfunction ϕ associated to $\mu_{1,p}$ is either positive or negative in Ω. In fact, if $\phi^+ \neq 0$ then, since ϕ^+ is also an eigenfunction associated to $\mu_{1,p}$, we get $\phi^+ > 0$ in $\overline{\Omega}$. Thus, $\phi^- = 0$ and ϕ is positive.

We show now the simplicity of $\mu_{1,p}$. To this purpose, for two positive eigenfunctions ϕ, ψ associated to $\mu_{1,p}$ consider the integral

$$I := \int_\Omega \left\{ |\nabla\phi|^{p-2}\nabla\phi\nabla\left(\frac{\phi^p - \psi^p}{\phi^{p-1}}\right) - |\nabla\psi|^{p-2}\nabla\psi\nabla\left(\frac{\phi^p - \psi^p}{\psi^{p-1}}\right)\right\}\mathrm{d}x.$$

Under the sole assumption that both $\phi, \psi \in W^{1,p}(\Omega)$ are positive and bounded in $\overline{\Omega}$ it follows that $I \geqslant 0$, and $I = 0$ only when $\psi = c\phi$ for a positive constant c. This is a consequence of the analysis in [11]. For the reader's benefit we sketch the argument. In fact,

$$\begin{aligned} I = {} & \int_\Omega (\phi^p - \psi^p)\big(|\nabla \log\phi|^p - |\nabla\log\psi|^p\big)\,\mathrm{d}x \\ & - \int_\Omega p\psi^p |\nabla\log\phi|^{p-2}(\nabla\log\phi)(\nabla\log\psi - \nabla\log\phi)\,\mathrm{d}x \\ & - \int_\Omega p\phi^p |\nabla\log\psi|^{p-2}(\nabla\log\psi)(\nabla\log\phi - \nabla\log\psi)\,\mathrm{d}x. \end{aligned}$$

Hence, by using the convexity of function $|x|^p$ with $p > 1$ ([11], inequality (4.1)) we infer that $I \geqslant 0$, and moreover $I = 0$ only when $\psi = c\phi$ for a positive constant c. Thus the simplicity of $\mu_{1,p}$ is proved.

The same argument implies that $\mu_{1,p}$ is the unique principal eigenvalue. In fact, suppose that ϕ is a positive eigenfunction associated to $\mu_{1,p}$ while $\mu \neq \mu_{1,p}$ is another eigenvalue which possesses a positive eigenfunction ψ. In this case, if we use $(\phi^p - \psi^p)/\phi^{p-1}$ as a test function in the equation for ϕ as an eigenfunction associated to $\mu_{1,p}$ and similarly employ $(\phi^p - \psi^p)/\psi^{p-1}$ in the equation for ψ then, after subtracting the resulting equalities, we arrive at

$$I = (\mu_{1,p} - \mu)\int_\Omega (\phi^p - \psi^p)\,\mathrm{d}x \geqslant 0.$$

However, $\mu > \mu_{1,p}$ and ϕ can be chosen greater than ψ in Ω. Since this contradicts the inequality, such an eigenvalue μ cannot exist.

To show the isolation of $\mu_{1,p}$ we follow the spirit of the corresponding statement in [1] (see also [12] for the case of the principal eigenvalue of (1.4) and $a = 1$), which we simplify in view of Lemma 8. Thus, assume on the contrary that there exists a sequence of eigenvalues $\mu_n \neq \mu_{1,p}$ with associated eigenfunction ϕ_n normalized by $\int_\Omega |\phi_n|^p = 1$ for all n, verifying $\mu_n \to \mu_{1,p}$. Notice that $\phi_n^\pm \neq 0$ for all n. Then, from the weak formulation of (1.2), we obtain

$$\int_\Omega \left(|\nabla \phi_n|^p + a|\phi_n|^p\right) \mathrm{d}x - \lambda \int_{\partial\Omega} |\phi_n|^p \,\mathrm{d}S = \mu_n.$$

By means of (2.5) we see that $|\phi_n|_{1,p}$ is bounded and so, passing to a subsequence, $\phi_n \rightharpoonup \phi_1$ weakly in $W^{1,p}(\Omega)$. It follows that ϕ_1 is a principal eigenfunction which can be assumed to be positive.

On the other hand, from the weak formulation of the equation satisfied by ϕ_n and by using ϕ_n^- as a test function, arguments similar as the ones employed in Lemma 8 show that

$$|\phi_n^-|_{1,p}^p \leqslant C \int_\Omega |\phi_n^-|^p \,\mathrm{d}x$$

for a positive constant C, not depending on n. Hence

$$\left|\{\phi_n < 0\}\right| \geqslant k > 0 \tag{2.10}$$

for some $k > 0$ and all n. However, since modulus a subsequence, $\phi_n \to \phi_1$ in $L^p(\Omega)$ and ϕ_1 is positive, Egorov's theorem implies that the uniform estimate (2.10) is not possible. Therefore, $\mu_{1,p}$ is isolated.

Finally, the features and asymptotic behavior of $\mu_{1,p}(\lambda)$ contained in statement (iv) can be shown by following the corresponding proof of Lemma 8 in [5]. □

Proof of Theorem 2. By using the terminology of Theorem 1, the key point is that σ is a principal eigenvalue of (1.4) if and only if

$$\mu_{1,p}(\sigma) = 0.$$

In view of property (iv) in Theorem 1 it is clear that (1.5) characterizes the existence of a zero of $\mu_{1,p}$ and so it characterizes the existence of a unique principal eigenvalue $\sigma := \sigma_{1,p}$ of (1.4) as well.

In addition

$$\int_\Omega \left(|\nabla \psi|^p + a|\psi|^p\right) \mathrm{d}x - \sigma \int_{\partial\Omega} |\psi|^p \,\mathrm{d}S = 0$$

if σ is a principal eigenvalue. Since $\lambda_{1,p}(a) > 0$ it follows that $\psi \neq 0$ on $\partial\Omega$ and so

$$\sigma_{1,p} = \frac{\int_\Omega (|\nabla \psi|^p + a|\psi|^p) \,\mathrm{d}x}{\int_{\partial\Omega} |\psi|^p \,\mathrm{d}S} \leqslant \frac{\int_\Omega (|\nabla u|^p + a|u|^p) \,\mathrm{d}x}{\int_{\partial\Omega} |u|^p \,\mathrm{d}S} \tag{2.11}$$

for all $u \in W^{1,p}(\Omega)$, $u \neq 0$ on $\partial\Omega$. Thus, $\sigma = \sigma_{1,p}$ also defines the first eigenvalue to (1.4). Relation (1.6) follows from the decreasing character of $\mu_{1,p}$ and the fact that $\lambda_{1,p}^{\mathcal{N}} = \mu_{1,p}(0)$.

The remaining assertions in Theorem 2 are consequences of Theorem 1. □

Remark 3. Inequality (2.11) states

$$\sigma_{1,p} = \inf_{u \in W^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega (|\nabla u|^p + a|u|^p)\,\mathrm{d}x}{\int_{\partial\Omega} |u|^p\,\mathrm{d}S}. \tag{2.12}$$

As already seen, such infimum is finite when $\lambda_{1,p}(a) > 0$. However, it can be checked that the infimum is $-\infty$ when $\lambda_{1,p}(a) \leqslant 0$ (details are omitted for brevity). This suggests setting $\sigma_{1,p} = -\infty$ in that case.

3. Existence and uniqueness

Our first objective is to prove the variational version of the method of sub- and super-solutions. For $p > 1$ we recall the notation $\varphi_p(t) = |t|^{p-2}t$.

Proof of Theorem 3. Following the ideas in [14], Theorem 2.4, we introduce the functional

$$J(u) = \int_\Omega \left\{ \frac{1}{p}|\nabla u|^p + \frac{a(x)}{p}|u|^p - F(x,u) \right\} \mathrm{d}x - \int_{\partial\Omega} G(x,u)\,\mathrm{d}S$$

with $F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t$ for $x \in \Omega$, $G(x,u) = \int_0^u g(x,t)\,\mathrm{d}t$ for $x \in \partial\Omega$, which we consider in the convex set

$$\mathcal{M} = \{ u \in W^{1,p}(\Omega)\colon\ \underline{u} \leqslant u \leqslant \overline{u} \text{ a.e. in } \Omega \}.$$

Notice that $\mathcal{M}$ defines a weakly closed subset of $W^{1,p}(\Omega)$. The functional J is sequentially lower semi-continuous and since both $\underline{u}$, $\overline{u}$ are bounded it is coercive in $\mathcal{M}$. Thus J achieves its infimum at some $u \in \mathcal{M}$ and we are showing that u is a weak solution to (1.7). For this, it is enough to show that $DJ[u](\varphi) \geqslant 0$ for every $\varphi \in C^1(\overline{\Omega})$.

To such proposal, for $\varepsilon > 0$ and arbitrary $\varphi \in C^1(\overline{\Omega})$ we set

$$\varphi_{\varepsilon,+} = (u + \varepsilon\varphi - \overline{u})^+, \qquad \varphi_{\varepsilon,-} = (\underline{u} - u - \varepsilon\varphi)^+$$

and observe that

$$u_\varepsilon := u + \varepsilon\varphi - \varphi_{\varepsilon,+} + \varphi_{\varepsilon,-} \in \mathcal{M}$$

for all $0 < \varepsilon < \varepsilon_0$. By taking the derivative of J at u in the direction of $u_\varepsilon - u$ we get

$$DJ(u)[u_\varepsilon - u] \geqslant 0.$$

This implies that,

$$\varepsilon DJ(u)[\varphi] \geqslant DJ(u)[\varphi_{\varepsilon,+}] - DJ(u)[\varphi_{\varepsilon,-}] \tag{3.1}$$

and we are showing next that

$$DJ(u)[\varphi_{\varepsilon,+}] \geqslant \rho(\varepsilon),$$

where $\rho(\varepsilon) = \mathrm{o}(\varepsilon)$ as $\varepsilon \to 0+$. In fact, since $DJ(\overline{u})[\varphi_{\varepsilon,+}] \geqslant 0$,

$$DJ(u)[\varphi_{\varepsilon,+}] \geqslant \bigl(DJ(u) - DJ(\overline{u})\bigr)[\varphi_{\varepsilon,+}]$$

and

$$\begin{aligned}
&\bigl(DJ(u) - DJ(\overline{u})\bigr)[\varphi_{\varepsilon,+}] \\
&\quad = \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla \varphi_{\varepsilon,+}\,\mathrm{d}x + \int_\Omega a(x)\bigl(\varphi_p(u) - \varphi_p(\overline{u})\bigr)\varphi_{\varepsilon,+}\,\mathrm{d}x \\
&\qquad - \int_\Omega \bigl(f(x,u) - f(x,\overline{u})\bigr)\varphi_{\varepsilon,+}\,\mathrm{d}x - \int_{\partial\Omega} \bigl(g(x,u) - g(x,\overline{u})\bigr)\varphi_{\varepsilon,+}\,\mathrm{d}S. \qquad (3.2)
\end{aligned}$$

By using the monotonicity of the p-Laplacian,

$$\begin{aligned}
&\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla \varphi_{\varepsilon,+}\,\mathrm{d}x \\
&\quad \geqslant \varepsilon \int_{\{\varphi_{\varepsilon,+}>0\}} \bigl(|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\bigr)\nabla \varphi\,\mathrm{d}x \\
&\quad \geqslant \varepsilon \int_{\{\varphi_{\varepsilon,+}>0\}\cap\{\overline{u}>u\}} \bigl(|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\bigr)\nabla \varphi\,\mathrm{d}x, \qquad (3.3)
\end{aligned}$$

since $\nabla u = \nabla \overline{u}$ almost everywhere in $\{u = \overline{u}\}$ [8]. Observe now that $|\{\varphi_{\varepsilon,+} > 0\} \cap \{\overline{u} > u\}| \to 0$ as $\varepsilon \to 0+$ and so the latter integral in (3.3) is $\mathrm{o}(\varepsilon)$ as $\varepsilon \to 0+$.

On the other hand, $|\varphi_{\varepsilon,+}| < \varepsilon|\varphi|$ in $\{\varphi_{\varepsilon,+} > 0\} \cap \{\overline{u} > u\}$. Hence,

$$\left| \int_\Omega \bigl(f(x,u) - f(x,\overline{u})\bigr)\varphi_{\varepsilon,+}\,\mathrm{d}x \right| \leqslant \varepsilon \int_{\{\varphi_{\varepsilon,+}>0\}\,\cap\,\{\overline{u}>u\}} \bigl|f(x,u) - f(x,\overline{u})\bigr||\varphi|\,\mathrm{d}x = \mathrm{o}(\varepsilon) \qquad (3.4)$$

as $\varepsilon \to 0+$. The remaining terms in (3.2) can be treated in the same way and so we achieve that,

$$DJ(u)[\varphi_{\varepsilon,+}] \geqslant \mathrm{o}(\varepsilon), \quad \varepsilon \to 0+.$$

A complementary argument shows that $DJ(u)[\varphi_{\varepsilon,-}] \leqslant \mathrm{o}(\varepsilon)$ as $\varepsilon \to 0+$. Therefore, (3.1) implies that

$$DJ(u)[\varphi] \geqslant 0$$

for arbitrary $\varphi \in C^1(\overline{\Omega})$. This means that u is a solution to (1.7). □

Remark 4. Theorem 3 can be extended to cover slightly more general settings. Namely, suppose that $\Omega \subset \mathbb{R}^N$ is smooth and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with Γ_1, Γ_2 disjoint $(N-1)$-dimensional closed manifolds. Consider the mixed problem

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = f(x,u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = g(x,u), & x \in \Gamma_1, \\ u(x) = h(x), & x \in \Gamma_2, \end{cases} \qquad (3.5)$$

with $h \in L^\infty(\Gamma_2)$. Then, under the extra condition

$$\underline{u} \leqslant h \leqslant \overline{u} \quad \text{on } \Gamma_2$$

and the hypotheses of Theorem 3 we achieve again a solution $u \in W^{1,p}(\Omega)$ to (3.5) lying between $\underline{u}$ and $\overline{u}$. The proof runs by the same lines of Theorem 3. As minor modifications, we have to take care of the condition $u = h$ on Γ_2 that must be incorporated to the definition of $\mathcal{M}$ and testing must be performed with functions $\varphi \in W^{1,p}(\Omega)$ vanishing on Γ_2.

As an immediate application of Theorem 3 we undertake the proof of Theorem 4.

Proof of Theorem 4. To prove the necessity of (1.8) we only consider, obviously, the case $\sigma_{1,p} > -\infty$. If a positive solution u to (1.1) exists then $u \neq 0$ on $\partial\Omega$. Otherwise,

$$-\Delta_p u + a\varphi_p(u) \leqslant 0$$

implies $u \leqslant 0$ in Ω if $u_{\partial\Omega} = 0$ (notice that $\sigma_{1,p}$ is finite if and only if $\lambda_{1,p}(a) > 0$). Thus, since $u \neq 0$ on $\partial\Omega$ we conclude that

$$\sigma_{1,p} \leqslant \frac{\int_\Omega (|\nabla u|^p + a|u|^p)\,\mathrm{d}x}{\int_{\partial\Omega} |u|^p\,\mathrm{d}S} < \lambda.$$

Assume now that $\lambda > \sigma_{1,p} \geqslant -\infty$. Let $\phi_1(\lambda)$ denote the principal positive eigenfunction satisfying $\sup_\Omega \phi_1(\lambda) = 1$. Then it can checked that $\underline{u} = A\phi_1(\lambda)$, $\overline{u} = B\phi_1(\lambda)$ define a sub-solution and a super-solution to (1.1) provided that

$$0 < A \leqslant (-\mu_{1,p})^{1/(r-p+1)}, \qquad B \geqslant \frac{(-\mu_{1,p})^{1/(r-p+1)}}{\inf \phi_1(\lambda)}.$$

Notice that a choice of A and B for all values of λ is possible when $\sigma_{1,p} = -\infty$. Thus, for suitable values of A and B we obtain, via Theorem 3, a positive solution to (1.1).

As for the uniqueness of a positive solution to (1.1) we first assert that all positive solutions $u \in W^{1,p}(\Omega)$ lie in $L^\infty(\Omega)$. In fact, observe that by setting $v = (u-k)^+$, $k > 0$, and employing v as a test function in the equation for u we arrive at

$$\int_\Omega \left(|\nabla v|^p + a(x)\varphi_p(u)v\right)\mathrm{d}x \leqslant |\lambda| \int_{\partial\Omega} \varphi_p(u)v\,\mathrm{d}S.$$

By adding to both sides of the inequality a term $M\int_\Omega \varphi_p(u)v$ with large enough M we get

$$|v|_{1,p}^p \leqslant C\left\{\int_\Omega \varphi_p(u)v\,\mathrm{d}x + \int_{\partial\Omega} \varphi_p(u)v\,\mathrm{d}S\right\}.$$

But such an estimate (see (2.2) and (2.3)) is just the starting point that leads to the boundedness of u if one proceeds as in Lemma 8. Thus $u \in L^\infty(\Omega)$. Notice in passing that the same argument works for the mixed problem (3.5) with $f = -u^r$, $g = \lambda\varphi_p(u)$ since the test function $v = (u-k)^+$ vanishes on Γ_2 provided that $k \geqslant |h|_\infty$.

Since a positive solution $u \in W^{1,p}(\Omega)$ is bounded, then $u \in C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(U_\eta)$ by the same reasons as those providing the smoothness of the eigenfunction ϕ_1 in Theorem 1. Hence, for two positive solutions u_1, u_2 to (1.1) we can consider the test functions $\varphi_1 = (u_1^p - u_2^p)/u_1^{p-1}$, $\varphi_2 = (u_1^p - u_2^p)/u_2^{p-1}$. With them we obtain the inequality (see [11])

$$I = \int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1 \nabla\varphi_1 - |\nabla u_2|^{p-2}\nabla u_2 \nabla\varphi_2\right) \mathrm{d}x \geqslant 0.$$

However, since

$$I = -\int_\Omega \left(u_1^{r-p+1} - u_2^{r-p+1}\right)\left(u_1^p - u_2^p\right) \mathrm{d}x,$$

then $u_1 = u_2$ is the unique option permitted by the former inequality. Thus, (1.1) admits a unique positive solution.

Regarding (iii), that $u_{r,\lambda}$ increases with λ is implied by the fact that $u_{r,\lambda}$ is sub-solution to (1.1) with λ replaced by $\lambda' \geqslant \lambda$. The uniqueness of positive solutions together with the existence, via [10], of uniform $C^{1,\beta}$ bounds of $u_{r,\lambda}$ when λ varies in bounded intervals, yield the continuous dependence of $u_{r,\lambda}$ with values in, say, $C^1(\overline{\Omega})$. Moreover, such continuity and the nonexistence of positive solutions for $\lambda = \sigma_{1,p}$ entail (1.9) when $\sigma_{1,p} > -\infty$.

To show (1.10), assume $\sigma_{1,p} = -\infty$, take $\lambda_n \to -\infty$ and set $u_n = u_{r,\lambda_n}$. From the equality

$$\int_\Omega \left(|\nabla u_n|^p + a u_n^p\right) \mathrm{d}x + (-\lambda_n)\int_{\partial\Omega} u_n^p \,\mathrm{d}S + \int_\Omega u_n^{r+1}\,\mathrm{d}x = 0,$$

together with the fact $0 < u_n \leqslant u_{n_0} \in L^\infty(\Omega)$ for $n \geqslant n_0$ we conclude, passing to a subsequence, that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, with $u \geqslant 0$. Since

$$(-\lambda_n)\int_{\partial\Omega} u_n^p \,\mathrm{d}S = \mathrm{O}(1),$$

we have $u = 0$ on $\partial\Omega$. By using test functions in $W_0^{1,p}(\Omega)$ in the weak formulation of the equation for u_n and passing to the limit, we see that u defines a solution to

$$-\Delta_p u + a\varphi_p(u) = -u^r$$

in Ω. When $\lambda_{1,p}(a) = 0$, this yields $u = 0$, so that $u_{r,\lambda} \to 0$ in $W^{1,p}(\Omega)$ as $\lambda \to -\infty$.

On the other hand, when $\lambda_{1,p}(a) < 0$ we obtain that $u > 0$ in Ω. In fact, let ϕ_n be the positive eigenfunction associated to $\mu_{1,p}(\lambda_n)$, normalized by $\sup_\Omega \phi_n = 1$. Then we have

$$\left\{-\mu_{1,p}(\lambda_n)\right\}^{1/(r-p+1)}\phi_n \leqslant u_n \quad \text{in } \Omega. \tag{3.6}$$

Next take α_n such that $\hat{\phi}_n = \alpha_n\phi_n$ verifies $|\hat{\phi}_n|_p = 1$ and observe that $\alpha_n \geqslant |\Omega|^{-1}$. We find that $\hat{\phi}_n \rightharpoonup \hat{\phi}$ weakly in $W^{1,p}(\Omega)$, where $\hat{\phi} > 0$ (indeed $|\hat{\phi}|_p = 1$). On the other hand, a careful analysis of the proof of Lemma 8 reveals that

$$\sup \alpha_n < \infty.$$

Hence we achieve, by passing to a subsequence if necessary,

$$\phi_n \rightharpoonup \frac{1}{\theta}\hat{\phi},$$

weakly in $W^{1,p}(\Omega)$, where $\theta := \overline{\lim}\,\alpha_n > 0$. Passing to the limit in (3.6), we finally obtain

$$\theta^{-1}\big(-\lambda_{1,p}(a)\big)^{1/(r-p+1)}\hat{\phi} \leqslant u$$

in Ω. Thus, $u > 0$, and it defines the unique positive solution to (1.11) when $\lambda_{1,p}(a) < 0$. By uniqueness, we obtain $u_n \to u$ weakly in $W^{1,p}(\Omega)$. This concludes the proof of (iii).

The proof of part (iv) will be included in the next section. □

4. Limit profiles

To prove Theorem 5 our first ingredient is a property on the maximum of solutions to (1.1) with varying r. The proof is based on a simple comparison argument.

Lemma 9. *For $r > p-1$ let $M_{r,\lambda} := \sup_\Omega u_{r,\lambda}$. Then $M_{r,\lambda}^{r-p+1}$ is an increasing function of r.*

Proof. Assume $r > q > p-1 > 0$. Then we clearly have

$$-\Delta_p u_{r,\lambda} + a\varphi_p(u_{r,\lambda}) = -{u_{r,\lambda}}^r \geqslant -M_{r,\lambda}^{r-q}{u_{r,\lambda}}^q \quad \text{in } \Omega,$$

while the boundary condition rests unchanged. It follows that the function

$$\overline{u} = M_{r,\lambda}^{(r-q)/(q-p+1)}u_{r,\lambda}$$

is a super-solution to problem (1.1) with r replaced by q. Since $\underline{u} = \varepsilon u_{q,\lambda}$ is a small enough sub-solution (for small ε) we obtain by uniqueness $\overline{u} \geqslant u_{q,\lambda}$. Thus $M_{r,\lambda}^{(r-p+1)/(q-p+1)} \geqslant M_{q,\lambda}$, which is the desired inequality. □

We can now proceed to prove Theorem 5.

Proof of Theorem 5. Let $v_r = u_{r,\lambda}/M_{r,\lambda}$. This function verifies

$$\begin{cases} -\Delta_p v(x) + av^{p-1}(x) = -M_{r,\lambda}^{r-p+1}v^r(x), & x \in \Omega, \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu}(x) = \lambda v^{p-1}(x), & x \in \partial\Omega, \end{cases} \tag{4.1}$$

and $|v_r|_\infty = 1$. Thanks to Lemma 9 we have $0 < M_{r,\lambda}^{r-p+1} \leqslant M_{p,\lambda}$, when $p-1 < r < p$, so that by the estimates in [10] we obtain that v_r is bounded in $C^{1,\beta}(\overline{\Omega})$ for certain $\beta \in (0,1)$. Thus for every sequence $r_n \to (p-1)+$ we may extract a subsequence, which will be relabeled as v_n, such that

$$v_n \to v$$

in $C^1(\overline{\Omega})$. We may also assume that

$$M_{r_n,\lambda}^{r_n-p+1} \to \theta$$

for some real number θ. Passing to the limit in the weak formulation of (4.1) we arrive at

$$\begin{cases} -\Delta_p v(x) + a v^{p-1}(x) = -\theta v^{p-1}(x), & x \in \Omega, \\ |\nabla v|^{p-2} \dfrac{\partial v}{\partial \nu}(x) = \lambda v^{p-1}(x), & x \in \partial\Omega, \end{cases}$$

with $v \geqslant 0$, $|v|_\infty = 1$ and thus $v > 0$ in $\overline{\Omega}$. Hence, thanks to the uniqueness assertion in Theorem 1 we have that

$$\theta = -\mu_{1,p}(\lambda),$$

while

$$v = \phi_1(\lambda),$$

where $\phi_1(\lambda)$ stands for the positive eigenfunction associated to $\mu_{1,p}$ with $\sup_\Omega \phi_1(\lambda) = 1$. It follows that $v_n \to \phi_1(\lambda)$ in $C^1(\overline{\Omega})$.

By writing

$$u_n = M_{r_n,\lambda} v_n = \bigl(-\mu_{1,p}(\lambda) + \mathrm{o}(1)\bigr)^{1/(r_n-p+1)} \bigl(\phi_1(\lambda) + \mathrm{o}(1)\bigr),$$

it is clear that assertions (a) and (b) follow immediately from the fact that $0 < -\mu_{1,p}(\lambda) < 1$ if $\lambda < \lambda^*$, provided λ^* exists (i.e., $\lambda_{1,p}(a) > -1$) while $-\mu_{1,p}(\lambda) > 1$ either if $\lambda > \lambda^*$ ($\lambda_{1,p}(a) > -1$) or for all λ ($\lambda_{1,p}(a) \leqslant -1$).

When $\lambda = \lambda^*$, we have $\mu_{1,p} = -1$, so that $M_{r,\lambda}^{r-p+1} \to 1$ as $r \to (p-1)+$. However, no further information on $M_{r,\lambda}$ is available from this convergence and a more subtle analysis is required.

Now, for technical reasons we restrict ourselves to the case of linear diffusion, that is, we consider $p = 2$. Multiplying (4.1) by ϕ_1 and integrating in Ω leads to

$$\int_\Omega \phi_1 \bigl(M_{r,\lambda}^{r-1} v_r^r - v_r\bigr)\,\mathrm{d}x = 0.$$

We may rewrite this equality as

$$\frac{M_{r,\lambda}^{r-1} - 1}{r-1} \int_\Omega \phi_1 v_r^r \,\mathrm{d}x = \int_\Omega \phi_1 v_r \frac{1 - v_r^{r-1}}{r-1}\,\mathrm{d}x. \tag{4.2}$$

Taking into account that $v_r \to \phi_1$ uniformly in $\overline{\Omega}$, and since $\phi_1 > 0$ in $\overline{\Omega}$, we obtain

$$v_r \frac{1 - v_r^{r-1}}{r-1} \to -\phi_1 \log \phi_1$$

uniformly in $\overline{\Omega}$ and hence, from (4.2),

$$\lim_{r\to 1+} \frac{M_{r,\lambda}^{r-1} - 1}{r-1} = -\frac{\int_\Omega \phi_1^2 \log \phi_1 \, \mathrm{d}x}{\int_\Omega \phi_1^2 \, \mathrm{d}x} = \log A, \tag{4.3}$$

where A is given by (1.14). Now, since from (4.3) we have

$$M_{r,\lambda} = \exp\left\{ \frac{1}{r-1} \log\bigl(1 + (\log A)(r-1) + \mathrm{o}(r-1)\bigr) \right\}$$

then we obtain

$$\lim_{r\to 1+} M_{r,\lambda} = A,$$

as was to be shown. The proof is finished. □

Now we deal with the limit as $r \to \infty$.

Proof of Theorem 6. Since $a = 0$ we consider the problem

$$\begin{cases} \Delta_p u(x) = u^r(x), & x \in \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu}(x) = \lambda u^{p-1}(x), & x \in \partial\Omega. \end{cases} \tag{4.4}$$

To obtain the asymptotic behavior of $u_{r,\lambda}$ as $r \to \infty$ we construct suitable sub- and super-solutions. To get a sub-solution we pick $\psi \in W^{1,p}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ the solution to

$$\begin{cases} -\Delta_p u(x) = 1, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.5}$$

The strong maximum principle [15] yields $\psi > 0$ in Ω while

$$c_1 \leqslant -|\nabla \psi|^{p-2} \frac{\partial \psi}{\partial \nu} \leqslant c_2 \quad \text{on } \partial\Omega$$

for some positive constants c_1, c_2.

We look for a sub-solution $\underline{u}$ under the form

$$\underline{u} = A(\psi + \gamma)^{-\alpha}, \quad \alpha = \frac{p}{r-p+1}, \tag{4.6}$$

where positive constants A, γ must be found. The condition

$$|\nabla \underline{u}|^{p-2} \frac{\partial \underline{u}}{\partial \nu} \leqslant \lambda \underline{u}^{p-1}$$

on $\partial\Omega$ is furnished by the choice $\gamma = \gamma_-$ with

$$\gamma_- = \left(\frac{c_2}{\lambda}\right)^{1/(p-1)}\alpha.$$

On the other hand, in order that $\underline{u}$ be a sub-solution it is required that

$$\alpha^{p-1}\big\{(p-1)(\alpha+1)|\nabla\psi|^p + (\psi+\gamma)\big\} \geqslant A^{r-p+1}$$

in Ω. Setting

$$\Phi = (p-1)|\nabla\psi|^p + \psi,$$

such inequality is satisfied if $A = A_-$ with

$$A_- = \alpha^{(p-1)/(r-p+1)}\Big(\inf_{\Omega}\Phi\Big)^{1/(r-p+1)}.$$

A super-solution of the form

$$\overline{u} = A_+(\psi+\gamma_+)^{-\alpha},$$

satisfying

$$\underline{u} \leqslant \overline{u}$$

in Ω is found by choosing the values:

$$\gamma_+ = \left(\frac{c_1}{\lambda}\right)^{1/(p-1)}\alpha, \qquad A_+ = \alpha^{(p-1)/(r-p+1)}\Big(2\sup_{\Omega}\Phi\Big)^{1/(r-p+1)},$$

provided that r is conveniently large (notice that $\gamma_+ \to 0$ as $r \to \infty$).

Finally, since

$$A_-\big(\psi(x)+\gamma_-\big)^{-\alpha} \leqslant u_{r,\lambda}(x) \leqslant A_+\big(\psi(x)+\gamma_+\big)^{-\alpha} \tag{4.7}$$

in Ω for large r we conclude that $u_{r,\lambda} \to 1$ uniformly in Ω as $r \to \infty$. $\Box$

Now we use the previous construction to conclude the proof of Theorem 4.

Proof of Theorem 4(iv). We first briefly discuss the existence of solutions to (1.12). Observe that the problem

$$\begin{cases} -\Delta_p u + au^{p-1} = -u^r, & x \in \Omega,\\ u = M, & x \in \partial\Omega,\end{cases}$$

has a unique positive solution $u = u_M \in C^{1,\beta}(\Omega)$ for every $M > 0$. In fact $\underline{u} = 0$, $\overline{u} = B\phi_1(\lambda_0)$ with $B > 0$ large can be used as a sub- and a super-solution provided $\mu_{1,p}(\lambda_0) < 0$. Uniqueness, which is achieved by the same ideas as in Theorem 1, implies that u_M is increasing with M.

On the other hand, local uniform $C^{1,\beta}$ bounds for u_M follow from the estimate

$$u_M \leqslant v_B, \quad x \in B$$

for every ball $B \subset \overline{B} \subset \Omega$, where $v = v_B$ is the minimal solution to

$$\begin{cases} -\Delta_p v(x) = |a|_\infty v^{p-1}(x) - v^r(x), & x \in B, \\ v = \infty, & x \in \partial B. \end{cases}$$

The existence of v_B is well documented (see, for instance, [13] and [7], Theorem 3). In conclusion,

$$u_M \to U$$

in $C^1(\Omega)$ where U defines a weak solution to (1.12) in the sense that $U \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$.

We now claim that, for fixed $r > p - 1$,

$$u_{r,\lambda} \to \infty$$

uniformly on $\partial\Omega$ as $\lambda \to \infty$. Since $u_M \leqslant u_{r,\lambda} \leqslant U$ in Ω for λ large we immediately achieve (1.13).

To show the claim we construct a suitable sub-solution $\underline{u}_\lambda$ to the auxiliary problem

$$\begin{cases} -\Delta_p u(x) + a u^{p-1}(x) = -u^r(x), & x \in U_\eta, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu}(x) = \lambda u^{p-1}(x), & x \in \partial\Omega, \\ u(x) = u_{r,\lambda}(x), & \operatorname{dist}(x, \partial\Omega) = \eta, \end{cases} \tag{4.8}$$

where $U_\eta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \eta\}$ and $\eta > 0$ is small. Notice that $u = u_{r,\lambda}$ is its unique solution (check once more the uniqueness proof in Theorem 1).

Following the preceding proof, a sub-solution of the form

$$\underline{u}_\lambda = A(\psi + \gamma)^{-\alpha},$$

with ψ and α as before, can be found in U_η by choosing

$$\gamma = \alpha \left\{ \frac{\sup_{\partial\Omega} |\nabla\psi|^{p-2}(-\partial\psi/\partial\nu)}{\lambda} \right\}^{1/(p-1)}$$

and taking $\lambda \geqslant \lambda_0$, $\eta \leqslant \eta_0$ and $0 < A \leqslant A_0$. Remark that

$$u_{r,\lambda} \geqslant u_{r,\lambda_0} \geqslant A\psi^{-\alpha} \geqslant \underline{u}_\lambda$$

on $\operatorname{dist}(x, \partial\Omega) = \eta$ for all $\lambda \geqslant \lambda_0$ provided $A < A_1$.

Now, by using $\overline{u}_\lambda = Bu_{r,\lambda}$, B large enough, as a super-solution, Theorem 3 (see Remark 4) implies in particular that

$$u_{r,\lambda} \geqslant \underline{u}_\lambda$$

for large λ. This shows the claim. □

Proof of Theorem 7. As observed in Theorem 4, sub- and super-solutions to (1.1) of the form $\underline{u} = A\phi_1(\lambda)$, $\overline{u} = B\phi_1(\lambda)$ can be found. Thus one arrives at

$$\bigl(-\mu_{1,p}(\lambda)\bigr)^{1/(r-p+1)}\phi_1(\lambda)(x) \leqslant u_{r,\lambda}(x) \leqslant \bigl(-\mu_{1,p}(\lambda)\bigr)^{1/(r-p+1)}\frac{\phi_1(\lambda)(x)}{\inf_\Omega \phi_1(\lambda)}$$

for all $r > p - 1$. This implies that

$$\varliminf_{r\to\infty} u_{r,\lambda}(x) \geqslant \phi_1(\lambda)(x), \quad x \in \Omega.$$

On the other hand, as in the proof of Theorem 6, a super-solution to (1.1) can be obtained in the form

$$\overline{u} = A\bigl(\psi(x) + \gamma\bigr)^{-\alpha},$$

with α, $\gamma = \gamma_+$ and ψ as in that proof, while A is chosen such that

$$A^{r-p+1} = 1 + |a|_\infty\Bigl(\sup_\Omega \psi + 1\Bigr)^p$$

for sufficiently large r. From the inequality $u_{r,\lambda} \leqslant \overline{u}$ one easily gets,

$$\varlimsup_{r\to\infty} u_{r,\lambda}(x) \leqslant 1.$$

A combination of these inequalities also gives

$$\lim_{r\to\infty} \sup_\Omega u_{r,\lambda} = 1.$$

To study the behavior of $\sup u_{r,\lambda}^{r-p+1}$ we first consider $a = 0$ in (1.1) but $p > 1$ arbitrary. In this case, inequality (4.7) directly leads to

$${u_{r,\lambda}}^{r-p+1}(x) \geqslant A_-^{r-p+1}\gamma_-^{-p}$$

on $\partial\Omega$. Since $\gamma_- \sim C\alpha$ as $r \to \infty$ such inequality says that

$$\lim_{r\to\infty} \sup_\Omega (u_{r,\lambda})^{r-p+1} = \infty. \tag{4.9}$$

To conclude with the case $a \in L^\infty(\Omega)$ arbitrary with λ large, we use an argument inspired in [3]. Let us begin assuming $a > 0$ in Ω and assume, arguing by contradiction, that $\sup u_{r,\lambda}^{r-p+1}$ is bounded. Choose $r_n \to \infty$ and set $u_n = u_{r_n,\lambda}$, $t_n = \sup u_n$, $u_n = t_n v_n$. Then v_n solves

$$\begin{cases} -\Delta_p v_n(x) + a v_n^{p-1}(x) = -u_n^{r_n-p+1} v_n^{p-1}(x), & x \in \Omega, \\ |\nabla v_n|^{p-2} \dfrac{\partial v}{\partial \nu}(x) = \lambda v_n^{p-1}(x), & x \in \partial\Omega. \end{cases}$$

Now, passing to a subsequence, $v_n^{r_n-p+1} \rightharpoonup h$ in $L^q(\Omega)$ for a nonnegative $h \in L^\infty(\Omega)$ and a conveniently large chosen $q > 1$. On the other hand, the estimates in [10] permit us showing that $v_n \to v$ in $C^{1,\gamma}(\overline{\Omega})$ where v is positive, $|v|_\infty = 1$ and solves

$$\begin{cases} -\Delta_p v(x) + a v^{p-1}(x) = -h v^{p-1}(x), & x \in \Omega, \\ |\nabla v|^{p-2} \dfrac{\partial v}{\partial \nu}(x) = \lambda v^{p-1}(x), & x \in \partial\Omega. \end{cases}$$

Since $0 < v(x) \leqslant 1$ in Ω and v is p-subharmonic it follows that $v(x) < 1$ for all $x \in \Omega$. Otherwise, $v = 1$ and from the equation $a + h = 0$ in Ω what is impossible. However, $v < 1$ implies $h = 0$ in Ω. Hence, v solves

$$\begin{cases} -\Delta_p v(x) + a v^{p-1}(x) = 0, & x \in \Omega, \\ |\nabla v|^{p-2} \dfrac{\partial v}{\partial \nu}(x) = \lambda v^{p-1}(x), & x \in \partial\Omega. \end{cases}$$

But this implies $\mu_1(\lambda) = 0$ which contradicts the existence of a positive solution to (1.1) (Theorem 1).

For an arbitrary $a \in L^\infty(\Omega)$, not necessarily positive let $u = \tilde{u}_{r,\lambda}$ be the solution to (1.1) with a replaced by $|a|_\infty > 0$ and notice that

$$u_{r,\lambda} \geqslant \tilde{u}_{r,\lambda}.$$

The conclusion follows from the fact that $\tilde{u}_{r,\lambda}$ satisfies (4.9). □

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References

[1] A. Anane, Simplicité et isolation de la première valeur propre du p-Laplacien avec poids, *C. R. Acad. Sci. Paris* **305**(I) (1987), 725–728.

[2] E.N. Dancer and Y. Du, On a free boundary problem arising from population biology, *Indiana Univ. Math. J.* **52** (2003), 51–67.

[3] E.N. Dancer, Y. Du and L. Ma, Asymptotic behavior of positive solutions of some elliptic problems, *Pacific J. Math.* **210** (2003), 215–228.

[4] J. García-Melián, J.D. Rossi and J. Sabina de Lis, A bifurcation problem governed by the boundary condition I, *Nonlinear Differ. Eq. Appl.* **14**(5,6) (2007), 499–525.
[5] J. García-Melián, J.D. Rossi and J. Sabina de Lis, Existence and uniqueness of positive solutions to elliptic problems with sublinear mixed boundary conditions, *Comm. Contemp. Math.* **11** (2009), 585–613.
[6] J. García-Melián, J.D. Rossi and J. Sabina de Lis, Layer profiles of solutions to elliptic problems under parameter-dependent boundary conditions, *Zeitschrift Anal. Anwend* **29** (2010), 1–17.
[7] J. García-Melián, J.D. Rossi and J. Sabina de Lis, Large solutions to an anisotropic quasilinear elliptic problem, *Ann. Math. Pure Appl.* **189** (2010), 689–712.
[8] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.
[9] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.
[10] G. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988), 1203–1219.
[11] P. Lindqvist, On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Proc. Amer. Math. Soc.* **109** (1990), 157–164.
[12] S. Martínez and J.D. Rossi, Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition, *Abstr. Appl. Anal.* **7** (2002), 287–293.
[13] J. Matero, Quasilinear elliptic equations with boundary blow-up, *J. d'Analyse Math.* **69** (1996), 229–247.
[14] M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, Berlin, 2008.
[15] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12**(3) (1984), 191–202.